## Introduction to MOSRP03

This has been a positive year for M-OSRP with significant progress on all projects. Noteworthy advances were reported on the two closely interrelated subprograms: (1) developing fundamental new effective capability to locate and delineate targets with both complex or simple reflector geometry without assumptions about the size of contrasts in properties and, furthermore, to achieve those objectives when the overburden cannot be adequately determined to allow current imaging-inversion methods to be effective; and, (2) developing methods to provide the necessary completeness of data collected and definition required by these non-linear sub-series: (i) data extrapolation and interpolation, (ii) source signature estimation and (iii) deghosting.

Among the highlights are:
(1) a method was developed and tested providing a robust new wavelet estimation method only requiring pressure measurements along the towed streamer, Z. Guo;
(2) a procedure was derived that provides deghosted towed streamer data from pressure measurements, with tests showing encouraging results, J. Zhang;
(3) a data reconstruction procedure that seeks maximum fidelity was analyzed and evaluated for primaries and multiples, for use with subsequent free-surface multiple attenuation, F. Miranda;
(4) the careful analysis, testing and evaluation of the 1D prestack leading order imaging series showed encouraging and reasonable effectiveness with absence of low vertical wave number information, passing a critical practical test, S. Shaw;
(5) following the 1D prestack results of Simon Shaw, Fang Liu developed and tested a 2D inverse series for a laterally and vertically varying subsurface, and first tests demonstrate moving the image of the target towards accurate spatial location with an input constant and unchanged velocity. This test was completed on our new IBM Blade Server, F. Liu;
(6) good progress and careful analysis for the modelling and inversion of headwaves for inclusion in imaging and as an internal multiple subevent, thereby expanding the class of events removed as internal multiples, B. Nita;
(7) extension of multi-parameter acoustic non-linear direct inversion to multi-parameter elastic inversion unambiguously defines the PP, PS, SP and SS data as minimum required
for that step beyond linear, and progress in analysis aimed at choosing elastic parameters to facilitate task separated interpretation and subseries for reflector location and parameter estimation in the elastic earth, H. Zhang;
(8) Kris Innanen brings us a step towards further realism of the subsurface model by including absorption in the forward and inverse series developing linear and non-linear series terms, coupled task subseries, and progress towards the ultimate objective of adding the task of achieving accurate Q compensation directly in terms of an inadequate Q estimate through non-linear communication between different events in the recorded data.

A five month SEG DL tour, provided an opportunity for me to visit and communicate with petroleum geophysicists around the world. That communication encouraged M-OSRP to consider returning to, and further pursuing, our earlier research in the area of multiples. Several actions followed:
(1) a M-OSRP Forum on Multiples was held at UH on Nov. 14, 2003 with attendees from 25 companies. Bill Dragoset, Ken Matson and I gave presentations on the Industry Status, Open Issues and Plans and DVD of the talks were distributed to sponsors.
(2) Progress on a SEG reprint volume on Multiple Attenuation, co edited by Bill Dragoset and myself, allowed an opportunity to provide a new perspective on the extensive literature on that subject; a preview of the soon-to-be-completed chapter introductions is provided in this report.
(3) A decision to expand M-OSRP into the area of multiples starting in 2004, with a team directed by Kris Innanen, and consulting by Dennis Corrigan and guidance from Adjunct Professor Ken Matson, of BP, moving M-OSRP into further analysis and an extension of inverse scattering internal multiple concepts. Activities will include the production and testing of algorithms and research prototype codes for 2D and then 3D marine towed-streamer data. The research of Bogdan Nita will interface and impact the analysis of expanded internal multiple efficacy. The plan is to add two Ph.D. graduate students to this project during the Spring 2004 semester. The delivery of M-OSRP extrapolation/interpolation methods will be coordinated with corresponding internal multiple codes.

A warm welcome to new Ph.D. graduate students Adriana Citlali and Sameera Rajapakshe. Einar Otnes, 2004 Statoil Fellow, is pursuing his Ph.D. with Bjorn Ursin at the Norwegian University of Science and Technology, Trondheim, and spending 2004 with M-OSRP. Einar is a creative, productive and capable young scientist and positive influence on our entire team. A very warm welcome to Brian Schlottmann, from Princeton University, the Amerada Hess Postdoctoral Fellow. Brian energetically launched into a research project on the analysis of diving waves and imaging and internal multiple subevents. Thanks to ConocoPhillips for graduate fellowships to Simon Shaw, Haiyan Zhang and Fang Liu. The contributions of our Adjunct Professors: Doug Foster, Bob Keys, Jacques Leveille, Ken Matson, Jon Sheiman, Bob Stolt, Hing Tan and Dan Whitmore to the guidance and mentoring of our students is gratefully acknowledged.

In summary, the 2003 progress to report is positive and significant, both in terms of critical robustness evaluation and extension of fundamentally new imaging and inversion capability, and also in the critically important practical prerequisites required to allow these new concepts to reach their potential. The planned M-OSRP growth into new practical and effective demultiple methods, dovetails with and serves the central thrust of our program for a fundamentally new vision and capability for processing primaries and multiples.

Arthur B. Weglein

University of Houston
March 24, 2004.

# Deghosting of towed streamer and ocean bottom cable data 

Jingfeng Zhang and Arthur B. Weglein


#### Abstract

We present the test results of the deghosting algorithm given by Weglein et al. (2002). For the towed streamer case, when the wavelet is available, the algorithm works well as expected. When the wavelet is not available, an approximate one is obtained using the method was provided by Zhang and Weglein (2003). The deghosting algorithm still works fine using the approximate wavelet, especially when the rough duration of the source wavelet is known. For the ocean bottom case, both the source wavelet and the hydrophone measurements $(P)$ are assumed to be known. The deghosting is performed using $P$ and its derivative $\left(\frac{d P}{d z}\right)$, which is calculated from $P$ and the source wavelet using the triangle relationship among these three quantities (Amundsen, 1995; Weglein et al., 2002). Numerical tests of this procedure for ocean bottom case is underway.


## 1 Introduction

Deghosting plays an important role in seismic exploration as a pre-requisite for many data processing techniques. For the towed streamer case, although some conventional procedures may still work without deghosting, accurate deghosting is necessary for some new techniques such as imaging without velocity and non-linear inversion. For the ocean bottom case, deghosting is required even for conventional methods since the ghost notch comes very early in the spectrum. Although both the field and its derivative are measured on the ocean bottom, deghosting remains a serious problem due to for instances the coupling between the geophone and the hydrophone or the noise in the geophone data.
Last year we presented the initial test of the deghosting algorithm given by Weglein et al. (2002). The test was focused on a single frequency and an idealized model. This year we apply the deghosting algorithm to a more realistic model and the result is presented in time domain.

In the following, we will briefly review the deghosting procedure for both towed streamer and ocean bottom cases, then we will give the numerical test results followed by conclusions.

## 2 Theory

The deghosting formula we use is (Weglein et al., 2002):

$$
\begin{equation*}
P^{\text {deghosted }}\left(\mathbf{r}, \mathbf{r}_{s}, \omega\right)=\int_{M . S .}\left(P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \frac{\partial G_{0}^{+}\left(\mathbf{r}, \mathbf{r}^{\prime}, \omega\right)}{\partial \mathbf{n}^{\prime}}-G_{0}^{+}\left(\mathbf{r}, \mathbf{r}^{\prime}, \omega\right) \frac{\partial P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)}{\partial \mathbf{n}^{\prime}}\right) \cdot d \mathbf{S}^{\prime} \tag{1}
\end{equation*}
$$

where M.S. denotes the measurement surface. In the following we will derive equation (1) and some related formulas. Start with the wave equation in the frequency domain

$$
\begin{equation*}
\nabla^{\prime^{2}} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)+\frac{\omega^{2}}{c^{2}\left(\mathbf{r}^{\prime}\right)} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)=A(\omega) \delta\left(\mathbf{r}^{\prime}-\mathbf{r}_{s}\right) \tag{2}
\end{equation*}
$$

where $A(\omega)$ is the source wavelet. Substituting $\frac{\omega^{2}}{c^{2}\left(\mathbf{r}^{\prime}\right)}$ with $\frac{\omega^{2}}{c_{0}^{2}}\left(1-\alpha\left(\mathbf{r}^{\prime}\right)\right)$, we find

$$
\begin{equation*}
\nabla^{\prime 2} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)+\frac{\omega^{2}}{c_{0}^{2}} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)=A(\omega) \delta\left(\mathbf{r}^{\prime}-\mathbf{r}_{s}\right)+\frac{\omega^{2}}{c_{0}^{2}} \alpha\left(\mathbf{r}^{\prime}\right) P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \tag{3}
\end{equation*}
$$

where $\alpha\left(\mathbf{r}^{\prime}\right)$ represents the difference between the actual medium and the reference medium, water. We additionally require the differential equation for Green's function in the reference medium:

$$
\begin{equation*}
\nabla^{\prime^{2}} G_{0}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right)+\frac{\omega^{2}}{c_{0}^{2}} G_{0}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right)=\delta\left(\mathbf{r}^{\prime}-\mathbf{r}\right) \tag{4}
\end{equation*}
$$

Of course, the Green's function $G_{0}$ for this differential equation is not unique. Its boundary conditions are versatile, and different choices lead to different applications/formulas, as we will see in a moment.
$P$ times Equation (4) minus $G_{0}$ times Equation (3) followed by a integral on certain volume gives

$$
\begin{align*}
& \int_{V}\left[P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \nabla^{\prime^{2}} G_{0}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right)-G_{0}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right) \nabla^{\prime 2} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)\right] d \mathbf{r}^{\prime} \\
= & \int_{V} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \delta\left(\mathbf{r}^{\prime}-\mathbf{r}\right) d \mathbf{r}^{\prime}-\int_{V} G_{0}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right)\left[A(\omega) \delta\left(\mathbf{r}^{\prime}-\mathbf{r}_{s}\right)+\frac{\omega^{2}}{c_{0}^{2}} \alpha\left(\mathbf{r}^{\prime}\right) P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)\right] d \mathbf{r}^{\prime} . \tag{5}
\end{align*}
$$

Using Green's second identity, the LHS of the equation above can be reduced to an integral on the surface of volume V:

$$
\begin{align*}
& \int_{V}\left[P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \nabla^{\prime 2} G_{0}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right)-G_{0}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right) \nabla^{\prime^{2}} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)\right] d \mathbf{r}^{\prime}  \tag{6}\\
= & \oint_{S}\left[P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \nabla^{\prime} G_{0}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right)-G_{0}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right) \nabla^{\prime} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)\right] \cdot d \mathbf{S}^{\prime} \tag{7}
\end{align*}
$$



Figure 1: Various volume configurations

Then we have:

$$
\begin{align*}
& \oint_{S}\left[P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \nabla^{\prime} G_{0}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right)-G_{0}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right) \nabla^{\prime} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)\right] \cdot d \mathbf{S}^{\prime} \\
= & \int_{V} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \delta\left(\mathbf{r}^{\prime}-\mathbf{r}\right) d \mathbf{r}^{\prime}-\int_{V} G_{0}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right)\left[A(\omega) \delta\left(\mathbf{r}^{\prime}-\mathbf{r}_{s}\right)+\frac{\omega^{2}}{c_{0}^{2}} \alpha\left(\mathbf{r}^{\prime}\right) P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)\right] d \mathbf{r}^{\prime} . \tag{8}
\end{align*}
$$

We will make repeated use of this formula. Choosing different volumes V, different boundary conditions for $G_{0}$ (as mentioned) and different positions $\mathbf{r}$, which can be either inside or outside volume, we can arrive at several different useful formulas.

If the whole half space below the M.S. is regarded as the volume V (Fig $1(\mathrm{a})$ ), position $\mathbf{r}$ is outside V, and a causal Green's function is chosen, then we have

$$
\begin{align*}
& P^{\text {deghosted }}\left(\mathbf{r}, \mathbf{r}_{s}, \omega\right) \\
= & \int_{\text {M.S. }}\left[P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \frac{\partial}{\partial z^{\prime}} G_{0}^{+}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right)-G_{0}^{+}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right) \frac{\partial}{\partial z^{\prime}} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)\right] d S^{\prime} . \tag{9}
\end{align*}
$$

This is the deghosting formula we will use. Note that in the above derivation, $\alpha$ is regarded as $\alpha_{\text {air }}, \alpha_{\text {water }}$ and $\alpha_{\text {earth }}$ and only $\alpha_{\text {earth }}$ is inside V .
If the configuration in Fig $1(\mathrm{~b})$ is used, where V horizontally extends to infinity, we have:

$$
\begin{equation*}
P\left(\mathbf{r}, \mathbf{r}_{s}, \omega\right)=A(\omega) G_{0}^{D D}\left(\mathbf{r}, \mathbf{r}_{s}, \omega\right)+\int_{M . S .} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \frac{\partial}{\partial z^{\prime}} G_{0}^{D D}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right) d S^{\prime} \tag{10}
\end{equation*}
$$

where $G_{0}^{D D}$ vanishes both at the free surface (denoted as F.S.) and the M.S.. The superscript "DD" indicates the use of double Dirichlet boundary conditions. This is the field prediction formula first given by Tan (1992). It has also been used as the wavelet estimation formula by Osen et al. (1998), who claim that the wavelet of a point source can be calculated using the field measurements on the M.S. and one extra measurement between the F.S. and the M.S.

Taking the derivative with respect to $z$ of both sides of Eq.(10) will give the derivative of the field prediction formula:

$$
\begin{equation*}
\frac{\partial P\left(\mathbf{r}, \mathbf{r}_{s}, \omega\right)}{\partial z}=A(\omega) \frac{\partial G_{0}^{D D}\left(\mathbf{r}, \mathbf{r}_{s}, \omega\right)}{\partial z}+\int_{M . S .} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \frac{\partial^{2} G_{0}^{D D}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right)}{\partial z^{\prime} \partial z} d S^{\prime} \tag{11}
\end{equation*}
$$

If we choose configuration Fig 1(c) instead, the scattered field prediction formula gives

$$
\begin{align*}
& \int_{M . S .}\left[P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \frac{\partial}{\partial z^{\prime}} G_{0}^{D}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right)-G_{0}^{D}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right) \frac{\partial}{\partial z^{\prime}} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)\right] d S^{\prime} \\
& =P\left(\mathbf{r}, \mathbf{r}_{s}, \omega\right)-A(\omega) G_{0}^{D}\left(\mathbf{r}, \mathbf{r}_{s}, \omega\right) \\
& =P_{s}\left(\mathbf{r}, \mathbf{r}_{s}, \omega\right) \tag{12}
\end{align*}
$$

where $G_{0}^{D}$ is the Green's function that vanishes on the F.S.
The wavelet estimation formula (Weglein and Secrest (1990)) can be derived if the configuration of Fig $1(\mathrm{~d})$ is used

$$
\begin{align*}
& \int_{M . S .}\left[P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \frac{\partial}{\partial z^{\prime}} G_{0}^{D}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right)-G_{0}^{D}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right) \frac{\partial}{\partial z^{\prime}} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)\right] d S^{\prime} \\
& =-A(\omega) G_{0}^{D}\left(\mathbf{r}, \mathbf{r}_{s}, \omega\right) . \tag{13}
\end{align*}
$$

This equation describes the triangle relationship among the source wavelet $A(\omega)$, the wave field $P$ and its derivative $\frac{d P}{d z}$. Any one of these three quantities can be calculated if the other two are known.

As discussed in (Zhang and Weglein, 2002), deghosting can be achieved either using one measurement on two surfaces or two measurements on one surface. The issue of the difference between the deghosting formula we used and conventional up-down separation is discussed in Appendix A.

The deghosting formula Eq.(1) computes the receiver side up-going field. The same integral on the source side will get rid of the source ghost; this is discussed further in Appendix B.

### 2.1 Towed streamer deghosting

In Eq. (1), both $P$ and its derivative on M.S. are needed. However, in practice, only $P$, the hydrophone measurement, is available. Our procedure is to use Eqs. (10) and (11) to predict the $P$ and its derivative on a new surface between the F.S. and the M.S.. When the wavelet is available, the prediction is exact.

As pointed out by Tan (1999), $G_{0}^{D D}$ in Eqs. (10) and (11) has a special property. It will vanish rapidly when the offset increases, for frequencies less than 120 Hz if water speed is $1500 \mathrm{~m} / \mathrm{s}$ and cable depth is 6.0 m . Hence for large offsets:

$$
\begin{gather*}
P\left(\mathbf{r}, \mathbf{r}_{s}, \omega\right) \approx \int_{M . S .} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \frac{\partial G_{0}^{D D}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right)}{\partial z^{\prime}} d S^{\prime}  \tag{14}\\
\frac{\partial P\left(\mathbf{r}, \mathbf{r}_{s}, \omega\right)}{\partial z} \approx \int_{M . S .} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \frac{\partial^{2} G_{0}^{D D}\left(\mathbf{r}^{\prime}, \mathbf{r}, \omega\right)}{\partial z^{\prime} \partial z} d S^{\prime} \tag{15}
\end{gather*}
$$

In the absence of a wavelet estimate, we can use these approximations to perform deghosting.
Last year, we tested the deghosting algorithm both with and without the source wavelet for a single frequency and using an idealized model. It turns out that very good deghosting results may be achieved when the source wavelet is available.

However, an unsatisfactory result is produced when the approximate field and its derivative are used. Through our analysis, we found that the approximate derivative of the field is far from accurate at small offsets (Fig 2). It is clear that Eq. (15) is a good approximation for large offsets, but at small offsets results were poorer than expected. We attempted to numerically extrapolate from large to small offset, but found that small offset is not predictable due to the variability of $\frac{d P}{d z}$. We speculate that an extrapolator constrained by physics might perform better.

We found one way to successfully step into the small offset area. The idea is to approximate the wavelet so that the first rapid varying derivative of $G_{0}^{D D}$ term at small offsets in Eq. (11) can be included. How to find an approximate wavelet? We went back to the wave field prediction Eq. (10), which is exact everywhere. Based on the assumption that the scattered field is small compared to the direct wave at zero offset, we assume the integral of the total field to be equal to that of the direct wave, hence neglecting the contribution of the scattered field. That is, if we put the output position $\mathbf{r}$ at zero offset, then the integral part in Eq.


Figure 2: The red line is the exact $\frac{d P}{d z}$; the black line represents the approximated $\frac{d P}{d z}$ without using the source wavelet
(10) is

$$
\begin{align*}
& \int_{M . S .} P\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \frac{\partial G_{0}^{D D}\left(\mathbf{r}^{\prime}, \mathbf{r}^{\prime \prime}, \omega\right)}{\partial z^{\prime}} d S^{\prime} \\
= & \int_{\text {M.S. }}\left[A(\omega) G_{0}^{D}\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)+P_{s}\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right)\right] \frac{\partial G_{0}^{D D}\left(\mathbf{r}^{\prime}, \mathbf{r}^{\prime \prime}, \omega\right)}{\partial z^{\prime}} d S^{\prime} \\
\approx & A(\omega) \int_{M . S .} G_{0}^{D}\left(\mathbf{r}^{\prime}, \mathbf{r}_{s}, \omega\right) \frac{\partial G_{0}^{D D}\left(\mathbf{r}^{\prime}, \mathbf{r}^{\prime \prime}, \omega\right)}{\partial z^{\prime}} d S^{\prime}, \tag{16}
\end{align*}
$$

where we have neglected the integral contribution of the scattered field. Only the wavelet is unknown in the above formula, and we may approximate it this way. The wavelet approximation formula has been shown to work well in previous year's effort. Using the approximate wavelet, a much better deghosting result is achieved. That is, we achieve good deghosting results from the measurements of the wave field $P$ on the M.S. only.

This year, instead of restricting ourselves to a single frequency, we test the last year's wavelet approximation and deghosting algorithm for all necessary frequencies and then consider the results in the time domain.

### 2.2 Ocean Bottom deghosting

On the ocean bottom, both the wave field $P$ and its derivative $\frac{d P}{d z}$ are measured. In order to avoid using the measurement of the derivative of $P$ (which is troublesome), we use $P$ and the source wavelet to calculate it using the triangle relationship Eq. (13), as mentioned by Amundsen (1995) and Weglein et al. (2002). performing a Fourier transform with respect to $x$ on both sides of Eq. (13), an algebraic triangle relationship is produced:

$$
\begin{align*}
& \frac{\partial}{\partial z^{\prime}} P\left(k_{x}, z^{\prime}, x_{s}, z_{s}, \omega\right) \\
= & \frac{A(\omega) e^{i k_{x} x_{s}}\left(e^{-i k_{z} z_{s}}-e^{i k_{z} z_{s}}\right)-i k_{z} P\left(k_{x}, z^{\prime}, x_{s}, z_{s}, \omega\right)\left(e^{-i k_{z} z^{\prime}}+e^{i k_{z} z^{\prime}}\right)}{e^{-i k_{z} z^{\prime}}+e^{i k_{z} z^{\prime}}} \tag{17}
\end{align*}
$$

where $k_{z}=\sqrt{k^{2}-k_{x}^{2}}$ and $z^{\prime}$ is the cable depth. The above formula is for 2D data. The 3D version is straightforward and can be found in Amundsen (1995). With $P$ and its derivative, we can perform deghosting.

## 3 Numerical tests for towed streamer data

Using the Cagniard-de Hoop method, we generated the synthetic data for the following model (Fig 3): a F.S. overlies 300 m of water (wave speed $1500 \mathrm{~m} / \mathrm{s}$ ), below which is a homogeneous acoustic halfspace characterized by wave speed $2250 \mathrm{~m} / \mathrm{s}$. The density is constant. The source wavelet is a Ricker wavelet with a peak frequency of 25 Hz (Fig 4). The data for the towed streamer cable (at a depth of 6 m ) is generated.

First, we Fourier transform each trace into the frequency domain. We then process the frequencies from zero to 120 Hz . Only the positive frequency is necessary to deal with, since the signal in time domain is real and hence in the Fourier domain we may assume conjugate symmetry. For each frequency, we approximate the source wavelet $A(\omega)$, predict $P(x, z, \omega)$ and its derivative at a new M.S., and deghost in frequency domain. Finally we transform the deghosted field back into the time domain.

The data generated for the towed streamer extends from 0s to 2.5 s. The receiver interval is 1 m and the largest receiver offset is 1500 m . The seismic traces at zero offset and 1500 m are presented in Fig 5. The signal includes the direct wave, the primary, multiples from first to forth order and their related ghosts.

It is apparent that the direct wave dominates at zero offset. This is the assumption upon which our wavelet approximation is based. The approximate wavelet is compared with the exact one in Fig 6 (a). There is no visible difference between the exact wavelet and the approximate one except at large $t$. Incidentally, if we treat the zero offset trace as the direct wave, then we can also approximate the wavelet. Fig 6(b) shows that there is almost no difference between these two methods.


Figure 3: Synthetic model

With this approximate wavelet, the field and its derivative at depth $\mathrm{z}=5.9 \mathrm{~m}$ is calculated for each frequency using Eqs. (10) and (11). The deghosting results at several offsets at a depth of $\mathrm{z}=5.8 \mathrm{~m}$ are presented in Fig 7. The result at zero offset is poor since the approximate wavelet treats each event in the signal as part of the direct wave. This inaccurate wavelet will eventually affect each event in the deghosting result; for large offsets, it will cause new small oscillations in the signal. If the rough duration of the source wavelet is known then the late events in the approximate wavelet could be cut off and hence we could obtain the almost exact wavelet. Using the exact wavelet, the deghosting results are shown in Fig 8; even in this case, we can see that the extremely strong direct wave at zero offset has not been totally eliminated. We think this is due to numerical errors.

## 4 Conclusions

We test and develop the deghosting algorithm given by Weglein et al. (2002), for a towed streamer acquisition. The deghosting algorithm works well, as expected, when the source wavelet is known. Otherwise, an approximate wavelet is found to perform deghosting. The approximate wavelet could work very well if we knew the rough duration of the source wavelet.

The effectiveness of the proposed scheme is currently being tested for an ocean bottom (OBS) type acquisition.


Figure 4: Ricker wavelet with peak frequency 25Hz

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## Appendix A

We will show that when the prediction point $\mathbf{r}$ is brought down to the M.S., Eq. (1) gives the conventional up-down separation result. In 1D, the up-down separation is:

$$
\begin{equation*}
P^{u p}\left(z^{\prime}, z_{s}, \omega\right)=\frac{1}{2}\left[P\left(z^{\prime}, z_{s}, \omega\right)-\frac{1}{i k} \frac{d P\left(z^{\prime}, z_{s}, \omega\right)}{d z^{\prime}}\right] . \tag{18}
\end{equation*}
$$

The deghosting formula in Eq. (1) reduces to

$$
\begin{equation*}
P^{u p}\left(z, z_{s}, \omega\right)=P\left(z^{\prime}, z_{s}, \omega\right) \frac{d G^{+}\left(z, z^{\prime}, \omega\right)}{d z^{\prime}}-G^{+}\left(z, z^{\prime}, \omega\right) \frac{d P\left(z^{\prime}, z_{s}, \omega\right)}{d z^{\prime}} \tag{19}
\end{equation*}
$$

where, assuming $z^{\prime}>z>z_{s}>0$,

$$
\begin{aligned}
& G^{+}\left(z, z^{\prime}, \omega\right)=\frac{1}{2 i k} e^{i k\left(z-z^{\prime}\right)} \\
& \frac{d G^{+}\left(z, z^{\prime}, \omega\right)}{d z^{\prime}}=\frac{1}{2} e^{i k\left(z-z^{\prime}\right)} .
\end{aligned}
$$

At the limit $z \rightarrow z^{\prime}$,

$$
\begin{aligned}
& G^{+}\left(z, z^{\prime}, \omega\right)=\frac{1}{2 i k}, \\
& \frac{d G^{+}\left(z, z^{\prime}, \omega\right)}{d z^{\prime}}=\frac{1}{2},
\end{aligned}
$$

so

$$
P^{u p}\left(z^{\prime}, z_{s}, \omega\right)=\frac{1}{2}\left[P\left(z^{\prime}, z_{s}, \omega\right)-\frac{1}{i k} \frac{d P\left(z^{\prime}, z_{s}, \omega\right)}{d z^{\prime}}\right],
$$

which is Eq. (18) as desired.

## Appendix B

Here we show that the integral in Eq. (1), if carried out on the source side, will eliminate the source ghosts. If we switch the source and receiver positions in Fig 9 the same data would be recorded (see Eq. 2). The many-source experiment in Fig 10 will therefore produce the same results as that of Fig 11.

The source-side deghosting of Fig 10 is exactly the receiver-side deghosting of Fig 11 (i.e., the source ghosts of Fig 10 would be the receiver ghosts of Fig 11). So the source-side deghosting seen in Fig 10 is
$P^{\text {source deghosted }}\left(\mathbf{r}_{s}^{\prime}, \mathbf{r}_{g}, \omega\right)=\int_{\text {S.S. }}\left(P\left(\mathbf{r}_{s}, \mathbf{r}_{g}, \omega\right) \frac{\partial G_{0}^{+}\left(\mathbf{r}_{s}, \mathbf{r}_{s}^{\prime}, \omega\right)}{\partial \mathbf{n}}-G_{0}^{+}\left(\mathbf{r}_{s}, \mathbf{r}_{s}^{\prime}, \omega\right) \frac{\partial P\left(\mathbf{r}_{g}, \mathbf{r}_{s}, \omega\right)}{\partial \mathbf{n}}\right) \cdot d \mathbf{S}$,
where S.S. represents the source surface and $\mathbf{r}_{s}^{\prime}$ can be any point between the S.S. and the F.S. The formula above computes the source-side deghosted field measured at $\mathbf{r}_{g}$ due to source at $\mathbf{r}_{s}^{\prime}$; the latter may be brought down to $\mathbf{r}_{s}$ in Fig 9 in accordance with the results of Appendix A.

(a)

(b)

Figure 5: (a) Data received at (0,6.0). (b) Data received at (1500,6.0)

(a)

(b)

Figure 6: (a) Red solid: exact source wavelet. Blue dots: approximated source wavelet using equation (16). (b) Red solid: wavelet using equation (16), approximate source. Blue dots: Approximate source wavelet using zero trace only.


Figure 7: Red solid: exact up-going field. Blue dots: predicted up-going field using approximate source wavelet. (a) At (0,5.8). (b) At (400,5.8). (c) At (800,5.8). (d) At $(890,5.8)$.


Figure 8: (Red solid: exact up-going field. Blue dots: predicted up-going field using approximate source wavelet. (a) At (0,5.8). (b) At $(400,5.8)$. (c) At $(800,5.8)$. (d) At $(890,5.8)$

## F.S.

$$
\vec{r}_{s} \ldots
$$

$$
\overrightarrow{\boldsymbol{r}}_{g} \nabla
$$



Figure 9: Single point source and point receiver experiment
F.S.



Figure 10: Many source experiment

## F.S

$$
\nabla \nabla \nabla \nabla \nabla \nabla \nabla \nabla \nabla \nabla \nabla \nabla \nabla \nabla \nabla \nabla \nabla
$$

.


Figure 11: Single source experiment with an array of receivers.

# Wavelet estimation below towed streamers 

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#### Abstract

Weglein and Secrest (1990) present a method for computing the scattered wavefield between the measurement surface (M.S.) and free surface (F.S.), and the reference wavefield below the measurement surface, given both the pressure and its normal derivative along the cable. Osen et al. (1998) and Tan (1999) show that the wavelet due to an isotropic source can be determined from pressure on the measurement surface and an extra hydrophone between the measurement surface and the free surface. Tan (1999) observes that in practice it is possible to well-estimate the wavefield above single towed streamer for points not directly under the source. Using the Tan (1999) wavefield prediction the wavelet can in principle be estimated from only a single cable (Weglein et al., 2000). However, the integral required for wavelet estimation requires data along the cable including the region excluded by the Tan's prediction. An approach to addressing that problem is presented here that adopts a generalized inverse viewpoint to find a good approximation to the wavelet. Empirical tests indicate that the particular alteration produces an accurate and stable estimated wavelet. Plans include testing the method in conjunction with free-surface multiple removal and then providing a comparison to the current industry-standard energy minimization method. Tests will include examples with interfering primaries and multiples.


## 1 Introduction

In wave-theoretic multiple attenuation methods (e.g. Carvalho (1992), Weglein et al. (1997), Verschuur et al. (1992)), knowledge of the source wavelet is one of the requirements. The energy-minimization criterion is often applied in practice to estimate the wavelet. Current methods based on the energy minimization criterion have proven to be useful under many circumstances. However, under complex conditions, e.g., with interfering events and weak internal multiples proximal to weak subsalt primaries, experience suggests that the energy minimization criterion is too blunt an instrument for that degree of subtlety. This is the motivation for deriving new methods to provide the source wavelet.

The free-surface and internal multiple methods predict the time and amplitude of multiples without subsurface information. However, if the dimension of data acquisition is not consistent with the dimension of the subsurface, e.g. a 3D subsurface requires 3D acquisition. When that requirement is lacking, the inverse scattering free surface and internal
demultiple algorithm experiences a diminishment of its effectiveness. However over the past years, Shell, PGS/DELFT, and Statoil have presented talks at the EAGE and SEG Annual Meetings showing 3D acquisition and extrapolations. That means the amplitude effects, e.g., obliquity, wavelet, and deghosting are all now high priority. Results of the research in this report directly addresses one of these key amplitude requirements. If the requirements of these free-surface and interval multiple methods are satisfied, they can surgically remove multiples without damaging proximal or interfering primaries.

The industry trend towards complex and costly plays raises the bar of required effectiveness for wave theoretic multiple removal and imaging-inversion techniques, and the prerequisites, such as the wavelet, that need to be provided.

The research described here is to test and progress the development of a new source wavelet estimation algorithm that requires only the pressure on the cable. In the following, we will first discuss the Extinction Theorem. We then show how to predict normal derivatives of the wavefield above the measurement surface and finally use these predictions to obtain an estimate of the source wavelet below the measurement surface.

## 2 Extinction Theorem

The acoustic wave equation can be written in the following form in the frequency domain, where $\vec{r}^{\prime}$ is any point in a half space below the free surface, $\vec{r}_{0}$ is the source location, $A(\omega)$ is the source signature, $\omega$ is the angular frequency, $c$ is the actual velocity, and $P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right)$ is the pressure field.

$$
\begin{equation*}
\nabla^{2} P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right)+\frac{\omega^{2}}{c^{2}\left(\vec{r}^{\prime}\right)} P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right)=A(\omega) \delta\left(\vec{r}^{\prime}-\vec{r}_{0}\right) . \tag{1}
\end{equation*}
$$

Using scattering theory, the actual earth can be parameterized as a homogeneous reference medium with embedded reflectors. Hence we replace with $c$ with $c_{0}$,

$$
\begin{equation*}
\frac{1}{c^{2}\left(\vec{r}^{\prime}\right)}=\frac{1}{c_{0}^{2}}\left[1-\alpha\left(\vec{r}^{\prime}\right)\right], \tag{2}
\end{equation*}
$$

where $c_{0}$ is the reference medium velocity, and $\alpha\left(\vec{r}^{\prime}\right)$ is called the perturbation, which is used to characterize the difference between the actual and reference media. Considering the Green's function in a homogenous medium with Dirichlet boundary conditions at both the free surface and the measurement surface due to a point source at $\vec{r}$, such that

$$
\begin{equation*}
\nabla^{2} G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right)+\frac{\omega^{2}}{c_{0}^{2}} G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right)=\delta\left(\vec{r}-\vec{r}^{\prime}\right) \tag{3}
\end{equation*}
$$

Applying Green's theorem to equations $P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right)$ and $G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right)$ :

$$
\begin{align*}
& \iint_{S} \mathrm{~d} s^{\prime}\left[P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right) \frac{\partial G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right)}{\partial n}-G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right) \frac{\partial P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right)}{\partial n}\right] \\
= & \iiint_{V} \mathrm{~d} \vec{r}^{\prime}\left[P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right) \nabla^{2} G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right)-G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right) \nabla^{2} P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right)\right] \tag{4}
\end{align*}
$$

Multiplying equation (3) by $P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right)$, and equation (1) by $G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right)$, and then substituting them into the right hand side of equation (4), we have

$$
\begin{array}{r}
\iint_{S} \mathrm{~d} s^{\prime}\left[P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right) \frac{\partial G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right)}{\partial n}-G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right) \frac{\partial P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right)}{\partial n}\right] \\
=\iiint_{V} \mathrm{~d} \vec{r}^{\prime}\left[P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right) \delta\left(\vec{r}-\vec{r}^{\prime}\right)\right]+\iiint_{V} \mathrm{~d} \vec{r}^{\prime}\left[-\frac{\omega^{2}}{c_{0}^{2}} G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right) \alpha\left(\vec{r}^{\prime}\right) P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right)\right] \\
+\iiint_{V} \mathrm{~d} \vec{r}^{\prime}\left[-A(\omega) G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right) \delta\left(\vec{r}^{\prime}-\vec{r}_{0}\right)\right] \tag{5}
\end{array}
$$

If we choose the integral volume $V$ to be between the free surface (F.S.) and the measurement surface (M.S.), then the second term on the right hand side of equation will be zero since the scatterer $\alpha\left(\vec{r}^{\prime}\right)$ (i.e. Earth) is outside of the integral volume $V$. We then choose $\vec{r}$ above M.S., and applying the sifting property of the delta function,

$$
\begin{equation*}
\iiint_{v} \mathrm{~d} \vec{r}^{\prime}\left[\delta\left(\vec{r}-\vec{r}^{\prime}\right) f\left(\vec{r}^{\prime}\right)\right]=f(\vec{r}) \tag{6}
\end{equation*}
$$

we have

$$
\begin{array}{r}
\iint_{S} \mathrm{~d} s^{\prime}\left[P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right) \frac{\partial G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right)}{\partial n}-G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right) \frac{\partial P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right)}{\partial n}\right] \\
=P\left(\vec{r}, \vec{r}_{0}, \omega\right)-A(\omega) G_{0}^{D D}\left(\vec{r}, \vec{r}_{0}, \omega\right) \tag{7}
\end{array}
$$

Finally, since we have chosen the Green's function $G_{0}^{\mathrm{DD}}\left(\vec{r}, \vec{r}^{\prime}, \omega\right)$ to satisfy Dirichlet boundary conditions on both F.S. and M.S., then

$$
\begin{equation*}
P\left(\vec{r}, \vec{r}_{0}, \omega\right)=A(\omega) G_{0}^{D D}\left(\vec{r}, \vec{r}_{0}, \omega\right)+\iint_{M S} \mathrm{~d} s^{\prime}\left[P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right) \frac{\partial G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right)}{\partial n}\right] \tag{8}
\end{equation*}
$$

where $\vec{r}$ is between F.S. and M.S.

## 3 Normal derivative

As shown in equation (8), $G_{0}^{D D}\left(\vec{r}, \vec{r}_{0}, \omega\right)$ is critical in order to compute the wavefield above M.S. It is a function of the frequency and the depth of the measurement surface. Tan (1999) discovered that $G_{0}^{D D}\left(\vec{r}, \vec{r}_{0}, \omega\right)$ is vanishingly small for typical marine streamer depths of approximately 6 m and seismic frequencies less than 125 Hz . Therefore, the first term on the right hand side of equation (8) can be ignored in comparison with the other terms. Also we choose the Green's function to satisfy Dirichlet boundary conditions on both F.S. and M.S., and we assume that the pressure at F.S. will be vanishing. This results in the key observation:

$$
\begin{equation*}
P\left(\vec{r}, \vec{r}_{0}, \omega\right) \approx \iint_{M S} \mathrm{~d} s^{\prime}\left[P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right) \frac{\partial G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right)}{\partial n}\right] \tag{9}
\end{equation*}
$$

For the normal derivative

$$
\begin{equation*}
\frac{\partial P\left(\vec{r}, \vec{r}_{0}, \omega\right)}{\partial n} \approx \iint_{M S} \mathrm{~d} s^{\prime}\left[P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right) \frac{\partial^{2} G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right)}{\partial z^{\prime} \partial z}\right] \tag{10}
\end{equation*}
$$

Equation (9) will be used to predict the wavefield above M.S. through an integral over the measurement surface. Once we obtain the Green's function, then equation (10) will produce normal derivatives directly.

## 4 Wavelet estimation

Since we require the normal derivatives under the source for wavelet estimation, we modify the idea of calculating the normal derivatives above the cable without dropping the wavelet term $A(\omega) G_{0}^{D D}\left(\vec{r}, \vec{r}_{0}, \omega\right)$ in equation (8). Hence

$$
\begin{equation*}
\frac{\partial}{\partial z} P\left(\vec{r}, \vec{r}_{0}, \omega\right)=A(\omega) \frac{\partial}{\partial z} G_{0}^{D D}\left(\vec{r}, \vec{r}_{0}, \omega\right)+\frac{\partial}{\partial z} \iint_{M S} \mathrm{~d} s^{\prime}\left[P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right) \frac{\partial G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right)}{\partial n}\right] \tag{11}
\end{equation*}
$$

We rewrite the wavelet estimation formula (Weglein and Secrest, 1990) as follows:

$$
\begin{equation*}
-A(\omega) G_{0}^{D}\left(\vec{r}_{b}, \vec{r}_{0}, \omega\right)=\iint_{M S} \mathrm{~d} s^{\prime}\left[P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right) \frac{\partial G_{0}^{D}\left(\vec{r}_{b}, \vec{r}^{\prime}, \omega\right)}{\partial n}-G_{0}^{D}\left(\vec{r}_{b}, \vec{r}^{\prime}, \omega\right) \frac{\partial P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right)}{\partial n}\right]( \tag{12}
\end{equation*}
$$

where the Green's function $G_{0}^{D}\left(\vec{r}_{b}, \vec{r}^{\prime}, \omega\right)$ satisfies the Dirichlet condition on the free surface and where $\vec{r}_{b}$ represents a location below M.S.
As the wavelet estimation equation (12) is the same as equation (8) on the cable (Weglein and Amundsen, 2003), we alternate the two equations by deliberately introducing some error.

In our case we choose the surface above the cable, and obtain the normal derivatives there, regarding them as the derivatives at cable, and then substitute them into the Weglein-Secrest equation (12). This we approximate

$$
\begin{equation*}
\frac{\partial}{\partial z} P\left(\vec{r}, \vec{r}_{0}, \omega\right) \approx \frac{\partial}{\partial z^{\prime}} P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right) \tag{13}
\end{equation*}
$$

which will be used to estimate the normal derivatives required in wavelet estimation formula (12).

Substituting equation (11) into equation (12), we can arrive at

$$
\begin{equation*}
A(\omega) \approx \frac{\iint_{S} \mathrm{~d} s^{\prime}\left[P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right) \frac{\partial G_{0}^{D}\left(\vec{r}_{b}, \vec{r}^{\prime}, \omega\right)}{\partial z}-G_{0}^{D}\left(\vec{r}_{b}, \vec{r}^{\prime}, \omega\right) \frac{\partial T\left(\vec{r}, \vec{r}_{0}, \omega\right)}{\partial z}\right]}{-G_{0}^{D}\left(\vec{r}_{b}, \vec{r}_{0}, \omega\right)+\iint_{M S} \mathrm{~d} s^{\prime} G_{0}^{D}\left(\vec{r}_{b}, \vec{r}^{\prime}, \omega\right) \frac{\partial G_{0}^{D D}\left(\vec{r}, \vec{r}_{0}, \omega\right)}{\partial z}} \tag{14}
\end{equation*}
$$

where

$$
\frac{\partial T\left(\vec{r}, \vec{r}_{0}, \omega\right)}{\partial z}=\frac{\partial}{\partial z} \iint_{S} \mathrm{~d} \vec{r}^{\prime}\left[P\left(\vec{r}^{\prime}, \vec{r}_{0}, \omega\right) \frac{\partial G_{0}^{D D}\left(\vec{r}, \vec{r}^{\prime}, \omega\right)}{\partial n}\right]
$$

The triangle relationship states that measured values of $P\left(\vec{r}, \vec{r}_{0}, \omega\right)$ and its normal derivative along a cable and $A(\omega)$ satisfy the exact equation (12). One might think equation (11), when evaluated on the cable, provides a second independent relationship that would allow to be directly determined from along the cable. However, Weglein and Amundsen (2003) demonstrate that these are the same relationship. If you temporarily ignore this fact, and substitute equation (11) into equation (12) to eliminate $\frac{\partial}{\partial n} P\left(\vec{r}, \vec{r}_{0}, \omega\right)$, then when $\vec{r}$ approaches cable, the expression in the denominator of equation (14) will be zero. The inverse is "unstable". To avoid this instability in the inversion, what is being suggested here is that values above the cable for $\frac{\partial}{\partial n} P\left(\vec{r}, \vec{r}_{0}, \omega\right)$ and $\frac{\partial}{\partial n} G_{0}^{D D}\left(\vec{r}, \vec{r}_{0}, \omega\right)$ are substituted for those at the cable in the integral to avoid the singularity. This has the effect of avoiding a singular division by solving a nearby perturbed problem with the anticipation that this will lead to a stable approximate solution.

## 5 Synthetic tests

We test the method in a homogenous model with three scatters. Figure 1 shows all estimated wavelet results with prediction surface changing from 5.3 m to 3.5 m from F.S. by equation (14). Figure 2 indicates the energy of the error with respect to the prediction surface and the ratio of the depth to wavelet. We can see that the least error location is at about 5.3 m from F.S, the ratio is about 0.02 . When you get closer to M.S., the error rapidly increases. This means the equation here approximates an unstable state. When it is far from the M.S., the error also increases, because the normal derivative is in greater error.


Figure 1: Wavelet estimation for different prediction depths. Wavelet at $z=0.7 \mathrm{~m}$ matches the input wavelet very closely.


Figure 2: Energy Error analysis. Left: Wavelet at $z=0.7 m$ has least error; the closer to M.S., the bigger the error gets, as it is close to unstable state. Right: Error vs. ratio of depth to wavelength.


Figure 3: Wavelet estimation with cable depth $\pm 15 \%$ error, the estimated wavelet is close to the input even if the cable depth has $\pm 15 \%$ error.

To test the stability of the cable depth, we assume the cable has $\pm 15 \%$ error. The result is shown in Figure 3. All of the three estimated wavelets are close to the actual wavelet. We further test synthetic datasets with water depth 40 m . The source is 2 m below the free surface, receivers are 6 m below the free surface, and the receiver interval is 2 m . Then equation (14) is used to estimate the wavelet (Figure 4). We compared three cases of weighting of the estimated wavelet. The weighting results are better then the averaged.

## 6 Conclusions

A method for estimating the wavelet directly from the data on a towed streamer was recently proposed by Weglein et al. (2002). That method proposed using the Tan (1999) wavefield prediction method to approximate the needed normal derivative along cable. However, the wavelet method requires an integral over all receivers for a given shot, and the Tan (1999) prediction is not accurate under the source. In this paper, we propose addressing this problem by not dropping the term, which is small only away from the source to achieve an algorithm that is valid for all offsets needed in the integral.
An intrinsic instability in this approach is addressed by seeking an approximate solution that replaces the unstable inversion by a "nearby" (i.e., perturbed) operation. Tests for different


Figure 4: Wavelet estimation using weighting: average means weighting by 1; $B$ and $C$ have different weighting.
prediction depths and noise stability on synthetic data are encouraging; further tests are planned for noise stability and the impact on wave-theoretic demultiple methods.

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# Seismic data reconstruction of primaries and multiples 

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#### Abstract

Seismic data reconstruction is an important practical prerequisite for the most complete "full wavefield" seismic processing algorithms (e.g., in multiple attenuation and imaging) that expect areal coverage on the measurement surface. Our objectives are to 1.) develop a clear and complete conceptual framework for data reconstruction of the entire wavefield at the measurement surface, that includes all seismic events (primaries and multiples), and 2.) develop practical algorithms for interpolation and extrapolation that adhere to this framework.

The most sophisticated data reconstruction techniques combine an inverse step followed by a forward data generation step. We show that the combination of making the inverse Born approximation followed by a forward Born-like data generation is applicable to both primaries and multiples, despite the single-scattering assumption of both the forward and inverse Born approximations.

We analyze an aperture-limited migration algorithm with constant background velocity for a 1D earth and study the ability to interpolate and extrapolate with different starting apertures in the pre-critical and post-critical zones. Data reconstruction with multiples is also tested, and the implications for free surface multiple removal are considered. We conclude that, with our approach, we can effectively reconstruct seismic data in the pre-critical region using data recorded at pre-critical offsets. Extrapolating from post-critical to pre-critical offsets confronts us with phase problems due to the reflectivity in the post-critical region. We show, for a simple example, that both primary and multiple reflections are reconstructed accurately in both time and amplitude.


## 1 Introduction and motivation

The search for hydrocarbons in ever more complex geological areas calls for more advanced seismic processing algorithms that make fewer and less rigid assumptions about a priori information relating to the subsurface (Weglein et al., 2003). These algorithms invariably require that seismic data be adequately sampled on the acquisition surface (in both the inline and crossline directions). The demands of these methods motivates research into new procedures for data interpolation and extrapolation.

One of the most challenging problems in marine exploration is incomplete data coverage. In particular the problem of missing data in the cross-line direction perpendicular to the
streamers is an impediment to effective removal of multiples and depth imaging of the data. For example, wave-theoretic multiple attenuation algorithms, such as the demultiple algorithms derived by Weglein et al. (1997) from the inverse scattering series, and other methods (e.g., Verschuur et al., 1992) are applicable to an unknown 3-D earth but expect data sampled on a grid at the surface to effectively attenuate all multiples. Extrapolation of near offsets is another important prerequisite for multiple attenuation and can be particularly challenging in shallow water where the first receiver records data in the post-critical regime. Wave-theoretic depth imaging algorithms (e.g., Claerbout, 1971; Stolt, 1978) also expect more complete data coverage than is typically acquired.

Although recent progress has been reported on the application of 3D free surface demultiple through a combination of acquisition and data reconstruction, it is recognized that the success of the combination critically depends on the effectiveness of the data extrapolation, and that is an area that could benefit from increased efficacy.

Different mapping operators have been developed, some simple and some complex, depending on their underlying assumptions and approximations. In general, the mapping operators can be classified into two groups: 1.) full seismic data reconstruction where the reflectivity depends on the incident angle (e.g., Stolt, 2002); and 2.) basic mapping reconstruction algorithms that are independent of incident angle.

In seismic data reconstruction, there are different components to be addressed. The fundamental idea of the proposed theory is a consecutive application of a Born-approximated migration, and demigration. The key elements of this approach are a weighted Kirchhofftype diffraction stack performed for depth migration using the inverse Born approximation, and a weighted Kirchhoff-type isochron stack integral for demigration using the forward Born approximation (Hubral et al., 1996; Santos et al., 2000). Although there are substantial differences between the inverse and the forward Born approximation, a combination of the two yields positive results regarding the seismic data reconstruction process. This is explained in the following section.

## 2 Forward and inverse scattering; A framework for data reconstruction

Forward scattering constructs the seismic wavefield, $D$, at any point in terms of reference medium propagation while inverse scattering inverts seismic data in terms of reference propagation. The forward scattering series is given by

$$
\begin{equation*}
D=G_{0} V G_{0}+G_{0} V G_{0} V G_{0}+\cdots \tag{1}
\end{equation*}
$$

where $G_{0}$ is the impulse response due to a point source in the reference medium and $V$ is the perturbation operator, which is the difference between the linear differential operators that describe wave propagation in the reference model and the actual model ( $L_{0}$ and $L$,
respectively). Events in time are difficult to predict in the forward series since the wave is propagating with the reference velocity. It takes an infinite sum to predict events at the correct time using the forward series. However, the forward series is an exact equation for primaries and multiples when it converges.
A widely-used approximation for inverting seismic data stems from the the assumption that, when $V$ is small in some sense, then the truncation of the series for the first term leads to

$$
\begin{equation*}
D \approx G_{0} V G_{0} \tag{2}
\end{equation*}
$$

This expression is the forward Born approximation and is not a good estimate for either primaries or multiples. It gives the incorrect time and a poor approximation to the amplitude of events.

To derive the inverse series, we consider the perturbation $V$ as an expansion

$$
\begin{equation*}
V=\sum_{n=1}^{\infty} V_{n}=V_{1}+V_{2}+V_{3}+\cdots \tag{3}
\end{equation*}
$$

where $V_{1}$ is the linear approximation to the earth properties. The inverse series is an infinite series in orders of the data on the measurement surface, $D_{0}$ (Weglein et al., 2003).

$$
\begin{equation*}
D_{0}=\Lambda_{g} \sum_{n=1}^{\infty}\left(G_{0} V\right)^{n} G_{0} \Lambda_{s}=\Lambda_{g}\left(G_{0} V G_{0}+G_{0} V G_{0} V G_{0}+\cdots\right) \Lambda_{s} \tag{4}
\end{equation*}
$$

where $\Lambda_{g}$ and $\Lambda_{s}$ are the projection of the field to the measurement surface. Introducing the parameter $\epsilon$ in (3), we will have $V=\sum \epsilon^{n} V_{n}$, and because we are on the measurement surface $D_{0} \Rightarrow \epsilon D_{0}$.

$$
\begin{equation*}
\epsilon D_{0}=\Lambda_{g}\left(G_{0} \epsilon V_{1} G_{0}+G_{0} \epsilon^{2} V_{2} G_{0}+\cdots+G_{0} \epsilon V_{1} G_{0} \epsilon V_{1} G_{0}\right) \Lambda_{s} \tag{5}
\end{equation*}
$$

The first order term is identified with $\epsilon$ of first order

$$
\begin{equation*}
\epsilon^{1}: D_{0}=\Lambda_{g} G_{0} V_{1} G_{0} \Lambda_{s} \tag{6}
\end{equation*}
$$

The second order terms contain $\epsilon^{2}$ coefficients

$$
\begin{equation*}
\epsilon^{2}: 0=\Lambda_{g} G_{0} V_{2} G_{0} \Lambda_{s}+\Lambda_{g} G_{0} V_{1} G_{0} V_{1} G_{0} \Lambda_{s} \tag{7}
\end{equation*}
$$

and the $n$th order terms contain $\epsilon^{n}$ coefficients

$$
\begin{equation*}
\epsilon^{n}: 0=\Lambda_{g} G_{0} V_{n} G_{0} \Lambda_{s}+\Lambda_{g} G_{0} V_{1} G_{0} V_{n-1} G_{0} \Lambda_{s}+\cdots+\Lambda_{g} G_{0} V_{1} G_{0} V_{1} G_{0} \cdots G_{0} V_{1} G_{0} \Lambda_{s} \tag{8}
\end{equation*}
$$

Equations (6)-(8) are the inverse scattering series and the same original reference operator $G_{0}$ is inverted at each step to solve for $V_{n}$. The inverse Born approximation is defined by

$$
\begin{equation*}
\epsilon^{1}: D_{0} \approx \Lambda_{g} G_{0} V G_{0} \Lambda_{s} \tag{9}
\end{equation*}
$$

As shown, $V_{1}$ is the portion of $V$ that is linear in the data.
The inverse series starts with data in time, hence it favors tasks with the reference velocity. By the nature of its terms, it has all the ingredients to work as an extrapolator, and its benefits are shown in the next section. From our understanding of the forward series, we might consider the first term in the inverse series as single scattering process, but the inverse series is a series in the data and this interpretation may cause some confusion.

Mathematically, the difference between the inverse Born approximation (9) and the forward Born approximation (2) is the substitution of $V_{1}$ in the first case, and $V$ in the second. The expression (9) is not only good for primaries and multiples, it is exact and precise without a thought or care about convergence. This understanding is at the heart of the first step of the inverse series as well as the engine for extrapolation. The success of seismic data reconstruction for multiples is based on the combination of inverse and forward. In some sense the demigration is the inverse task to migration, and by this rule it can handle multiples.

## 3 Aperture Migration, a 1D Example

### 3.1 Finite aperture migration

To invert for medium properties requires choosing the set of parameters that you want to invert for. The chosen set of parameters defines an earth model type, and the details of the data reconstruction will depend on that choice. The meaning of a model-type-independent task-specific subseries is that the defined task is achievable with precisely the same algorithm for an entire class of earth model types. From this consideration we conclude that seismic data reconstruction is model type dependent.

We start with a convenient reflection model, realistic at moderate reflection angles provided by perturbation theory. This model assumes that the reflections are caused by small and rapid deviations from a constant background. In this model every subsurface point can reflect seismic energy.

Under this assumption, we also assume that we are dealing with a pressure field described by the scalar wave equation. $G_{0}$ is the unit impulse response in a background of homogeneous velocity $c$, and $G$ is the unit impulse response for the actual inhomogeneous velocity $v$. Thus $G$ and $G_{0}$ obey the scalar wave equations

$$
\begin{equation*}
\left(\nabla^{2}-\frac{1}{v^{2}} \frac{\partial^{2}}{\partial t^{2}}\right) G=-\delta\left(t_{g}-t_{s}\right) \delta\left(x_{g}-x_{s}\right) \delta\left(z_{g}-z_{s}\right), \quad G=0 \quad t<0 \tag{10}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\nabla^{2}-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}\right) G_{0}=-\delta\left(t_{g}-t_{s}\right) \delta\left(x_{g}-x_{s}\right) \delta\left(z_{g}-z_{s}\right), \quad G_{0}=0 \quad t<0 \tag{11}
\end{equation*}
$$



Figure 1: An acoustic 1D earth model with two medium velocities. The first medium has $c=1500 \mathrm{~m} / \mathrm{s}$ and the second medium with $v=2250 \mathrm{~m} / \mathrm{s}$.

Here $\left(x_{g}, z_{g}\right)$ and $\left(x_{s}, z_{s}\right)$ are source and receiver locations respectively. $t=t_{g}-t_{s}$ is the travel time from source to receiver. By subtracting (10) from (11) we obtain

$$
\begin{equation*}
\left(\nabla^{2}-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}\right)\left(G-G_{0}\right)=-\left(\frac{1}{c^{2}}-\frac{1}{v^{2}}\right) \frac{\partial^{2}}{\partial t^{2}} G . \tag{12}
\end{equation*}
$$

Defining the difference between impulse responses, $P=G-G_{0}$, as the scattered wavefield, the solution for $P$ can be expressed as an integral equation

$$
\begin{align*}
P\left(x_{g}, z_{g} \mid x_{s}, z_{s} ; t_{s}-t_{g}\right)= & \int d x \int d z \int d t G_{0}\left(x_{g}, z_{g} \mid x, z ; t_{g}-t\right) \\
& \frac{A(z)}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} G\left(x, z \mid x_{s}, z_{s} ; t-t_{s}\right) . \tag{13}
\end{align*}
$$

where $A(z)$, shown in Figure 1, is the velocity perturbation and is defined as

$$
\begin{equation*}
A(z)=1-\frac{c^{2}}{v^{2}} \tag{14}
\end{equation*}
$$

$c$ is the velocity in the background medium and $v$ is the true velocity of our model. Here there is a combination of the transmission term $G_{0}$ which carries energy from source to reflector, the reflector response $A$, and the transmission term $G$ that carries energy from the reflector to the surface. $G$ is the perturbed field, and it contains the direct wave and all
multiple reflections. To linearize this equation, $G$ is replaced by $G_{0}$ using the first two terms of the Born series obtained from the Lippmann-Schwinger equation

$$
\begin{align*}
P\left(x_{g}, z_{g} \mid x_{s}, z_{s} ; t_{s}-t_{g}\right)= & \int d z \int d x \int d t G_{0}\left(x_{g}, z_{g} \mid x, z ; t_{g}-t\right) \\
& \times \frac{A(z)}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} G_{0}\left(x, z \mid x_{s}, z_{s} ; t-t_{s}\right) \tag{15}
\end{align*}
$$

Equation (15) corresponds to the Born approximation in (2). Physically the approximation of $G$ by $G_{0}$ means that all but primary reflections are neglected.

The solution to equation (11) in the frequency domain is given by

$$
\begin{gather*}
G_{0}\left(k_{g x}, k_{g z} \mid z ; \omega, t_{g}\right)=-e^{i\left(\omega t_{g}-k_{g x} x\right)} \frac{e^{-i k_{g z}\left|z_{g}-z\right|}}{2 i k_{g z}}  \tag{16}\\
G_{0}\left(k_{s x}, k_{s z} \mid x_{s}, z_{s} ; \omega, t_{s}\right)=-e^{i\left(\omega t_{s}-k_{s x} x\right)} \frac{e^{-i k_{s z}\left|z-z_{s}\right|}}{2 i k_{s z}} \tag{17}
\end{gather*}
$$

where

$$
\begin{equation*}
k_{g z}+k_{s z}=\operatorname{sgn}(\omega) \sqrt{\frac{\omega^{2}}{c^{2}}+k_{g x}^{2}}+\operatorname{sgn}(\omega) \sqrt{\frac{\omega^{2}}{c^{2}}+k_{s x}^{2}} \tag{18}
\end{equation*}
$$

Equation (15) becomes

$$
\begin{align*}
P\left(k_{g x}, 0 \mid k_{s x}, 0 ; \omega\right) & =-\int d x \int d z \frac{e^{-i k_{g z}|z|}}{2 i k_{g z}} e^{-i\left(k_{g x} x\right)} A(z) \frac{\omega^{2}}{c^{2}} \frac{e^{-i k_{s z}|z|}}{2 i k_{s z}} e^{i\left(k_{s x} x\right)}  \tag{19}\\
& =2 \pi \frac{\omega^{2}}{c^{2}} \frac{A\left(k_{z}\right)}{k_{z}^{2}} \tag{20}
\end{align*}
$$

where $k_{z}=k_{g z}+k_{s z}$. There are different ways to invert equation (20). We exploit the degree of freedom to compensate the aperture to a finite region by using a weight function $L\left(k_{z}, k_{g x}\right)$. It must carry the information of the aperture and satisfy the constraint

$$
\begin{equation*}
\int \frac{\omega^{2}}{k_{z}^{2}} \cdot 2 \pi \cdot L\left(k_{g x}, k_{z}\right) \cdot d k_{g x}=1 \tag{21}
\end{equation*}
$$

The migrated image is then described by

$$
\begin{equation*}
A(z)=\frac{1}{2 \pi} \int e^{-i k_{z} \cdot z} \cdot d k_{z} \int L\left(k_{g x}, k_{z}\right) \cdot d k_{g x} \int d x_{g} \cdot e^{-i k_{g x} \cdot x_{g}} \cdot P\left(x_{g}, c \cdot \sqrt{k_{z}^{2} / 4+k_{g x}^{2}}\right) . \tag{22}
\end{equation*}
$$

In other words

$$
\begin{equation*}
A(z)=\int d x_{g} \int d \omega \cdot I\left(z, x_{g}, \omega\right) \cdot P\left(x_{g}, \omega\right) \tag{23}
\end{equation*}
$$

where

$$
\begin{equation*}
I\left(z, x_{g}, \omega\right)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{\omega}{c} L\left(\frac{\omega}{c} \xi, 2 \frac{\omega}{c} \sqrt{1-\xi^{2}}\right) e^{-\frac{\omega}{c}\left(2 \sqrt{1-\xi^{2}} \cdot z+\xi \cdot x_{g}\right)} \frac{2}{c \sqrt{1-\xi^{2}}} d \xi \tag{24}
\end{equation*}
$$

and

$$
\begin{equation*}
\xi=\frac{c}{\omega} k_{g x} . \tag{25}
\end{equation*}
$$

In general, the integral (24) cannot be solved analytically. However, it may be simplified to an approximate ray-theoretical expression via the method of stationary phase, which provides a way of analyzing the main contributions. Reducing it to its basic structure, the integral in equation (24) can be written in the form

$$
\begin{equation*}
I(\omega)=\int_{-\infty}^{\infty} f(\xi) \cdot e^{i \omega q(\xi)} d \xi \tag{26}
\end{equation*}
$$

The method of stationary phase is based on the observation that for high frequencies, the factor $e^{i \omega q(\xi)}$ oscillates very rapidly, thus covering full periods in very small intervals of $\xi$, (for details, see appendix A).

If $f(\xi)$ is not itself an oscillating function, its values do not strongly vary in any such interval. Thus, the integration over a full period of $e^{i \omega q(\xi)}$ yields approximately zero and does not contribute to the overall value of the integral. The only regions where it does not oscillate are those where the phase function $q(\xi)$ remains approximately constant or stationary.

For aperture migration, the limits of integrations of expression (26) are going to change to finite values $a$ and $b$, because of finite data consideration, reducing to the integral

$$
\begin{equation*}
I(\omega)=\int_{a}^{b} f(\xi) \cdot e^{i \omega q(\xi)} d \xi \tag{27}
\end{equation*}
$$

The analysis of the migration integral by the stationary phase approximation yields

$$
\begin{equation*}
I(\omega) \simeq f\left(\xi_{s t}\right) e^{i \omega q\left(\xi_{s t}\right)} \sqrt{\frac{2 \pi}{-\omega q^{\prime \prime}\left(\xi_{s t}\right)}}+\frac{1}{i \omega}\left[\frac{f(b)}{q^{\prime}(b)} e^{i \omega q(b)}-\frac{f(a)}{q^{\prime}(a)} e^{i \omega q(a)}\right] \tag{28}
\end{equation*}
$$

where $\xi_{s t}$ is the stationary point of the phase $q=\tau_{D}-\tau_{R}$, that is the tangency point between the Huygens and the reflection traveltime curves. The first term of the expression (28) is the dominant part of the total migrated section, and the second terms comes from the endpoints of the integration/stacking operator.

### 3.2 Regularization by neutralizers

Using separation of variables, the aperture compensation function is described as the product of two functions: the first one carries geometrical information about the aperture and the second carries information about the phase (Stolt and Benson, 1986)

$$
\begin{equation*}
L\left(k_{x}, k_{z}\right)=S\left(\frac{k_{g x}}{k_{z}}\right) \cdot F\left(c \cdot \sqrt{k_{z}^{2} / 4+k_{g x}^{2}}\right) \tag{29}
\end{equation*}
$$

According to Bleistein and Handelsman (1986), a neutralizer is defined to be one if it is in the neighborhood of a stationary value and zero outside this region. For a regularization with a neutralizer function, we have

$$
\begin{equation*}
S\left(\frac{k_{g x}}{k_{z}}\right)=B\left(h_{1}, h_{2}, x_{g}-x_{s}\right) \cdot T\left(k_{g x} / k_{z}\right) \tag{30}
\end{equation*}
$$

where $T\left(k_{g x} / k_{z}\right)$ is a weight function that compensates for the amplitude and $B\left(h_{1}, h_{2}\right)$ is our neutralizer function defined (in the limits of the aperture, with $h_{1}$ as the starting aperture offset and $h_{2}$ final aperture offset) as

$$
B\left(h_{1}, h_{2}, x_{g}-x_{s}\right)= \begin{cases}1 & \left(x_{g}-x_{s}\right) \in\left(h_{1}, h_{2}\right)  \tag{31}\\ 0 & \text { otherwise }\end{cases}
$$

Defining $\beta=k_{g x} / k_{z}, \gamma=c k_{z} / 2$ and the evaluation of the end points of the aperture $\beta_{1}=$ $k_{x s t a t} / k_{z s t a t}=x_{g 1} / 4 z \beta_{2}=k_{x s t a t} / k_{z s t a t}=x_{g 2} / 4 z$ we can express (21) as

$$
\begin{equation*}
\int_{\beta_{1}}^{\beta_{2}} S(\beta) \cdot F\left(\gamma \cdot \sqrt{1+4 \beta^{2}}\right) \cdot \frac{1 / c^{2} \cdot \gamma^{2} \cdot\left(1+4 \beta^{2}\right)}{4 \gamma^{2} / c^{2}} \cdot 2 \pi \cdot \frac{d \beta}{2 \gamma / c}=1 \tag{32}
\end{equation*}
$$

where $F$ is an arbitrary function to be chosen. If we choose $F\left(\gamma \cdot \sqrt{1+4 \beta^{2}}\right)=\gamma \cdot \sqrt{1+4 \beta^{2}} / c$, then (21) becomes

$$
\begin{equation*}
\int S(\beta) \cdot\left(1+4 \beta^{2}\right)^{3 / 2} \cdot \frac{\pi}{4} \cdot d \beta=1 \tag{33}
\end{equation*}
$$

Finally, the expression for aperture compensation using a neutralizer is defined as

$$
\begin{equation*}
L\left(k_{g x}, k_{z}\right)=\frac{c}{\pi} \cdot \frac{\omega^{-1}}{\left(1+4 k_{g x}^{2} / k_{z}^{2}\right)^{1 / 2}} \cdot \frac{1}{\left(\beta_{2}-\beta_{1}\right)} . \tag{34}
\end{equation*}
$$

### 3.3 Regularization by taper functions

The neutralizer function can be replaced by a taper function only having partial derivatives up to second order (Sun, 1998, 2002). To be in accordance with the neutralizer function, the taper must be unity within a subaperture and be zero on the boundary of the subaperture, it must decrease smoothly from unity to zero.
Under this assumption,

$$
\begin{equation*}
B\left(h_{1}, h_{2}, x_{g}-x_{s}\right)=\frac{1}{2}\left(1+\cos \left[\pi\left(\frac{\beta-\beta_{1}}{\beta_{2}-\beta_{1}}\right)\right]\right) . \tag{35}
\end{equation*}
$$

According to (35) the aperture compensation function using a taper function is

$$
\begin{equation*}
L\left(k_{g x}, k_{z}\right)=\frac{1}{2 \pi}\left(1+\cos \left[\pi\left(\frac{k_{g x} / k_{z}-\beta_{1}}{\beta_{2}-\beta_{1}}\right)\right]\right) \frac{\omega^{-1}}{\left(1+4\left(k_{g x} / k_{z}\right)^{2}\right)^{1 / 2}} \frac{c}{\beta_{2}-\beta_{1}} . \tag{36}
\end{equation*}
$$

For our numerical examples, we use an acoustic earth model that consists of two semi-infinite homogeneous halfspaces. The upper halfspace is defined to have a velocity of $c(z<300 \mathrm{~m})=$ $1500 \mathrm{~m} / \mathrm{s}$, and the velocity is the lower halfspace is $v(z \geq 300 \mathrm{~m})=2250 \mathrm{~m} / \mathrm{s}$. Density is kept constant over the two halfspaces.


Figure 2: Model of a single interface. The velocity in the upper and lower media are $c=1500 \mathrm{~m} / \mathrm{s}$ and $v=2250 \mathrm{~m} / \mathrm{s}$, respectively, and density is constant.

## 4 Structure of the aperture migrated image

The aperture migrated image is given by

$$
\begin{equation*}
A(z)=\int d x_{g} \int d \omega \cdot\left(I_{0}+I_{1}+I_{2}\right) \cdot P\left(x_{g}, \omega\right) \tag{37}
\end{equation*}
$$

where there is a filtering process, done by the integral over $d \omega$, and also a stacking process, performed by the integral over offset. The three terms in the integrand correspond to contributions from the stationary phase calculation $\left(I_{0}\right)$ and each of the end points of the migration aperture ( $I_{1}$ and $I_{2}$ ). For a more detailed explanation, see Sun (1998) and Stolt and Benson (1986).

The finite large aperture migration results in a migrated imaged with three parts. One comes from the tangent point between the traveltime curves of the reflected and point-diffracted rays and gives a migrated signal (Jaramillo and Bleistein, 1997). The others are from the
endpoints of the migration aperture and result in migration noise as shown by Hertweck and Goertz (1999).

To guarantee the true-amplitude reconstruction as well as to attenuate migration noise, the migration aperture should be positioned so that its central part contains the tangent point. The true-amplitude weight function should be modified so it tapers the input data near the boundary of the migration aperture.

Depending on where the stationary point is located there are different contributions in the stationary phase form and to the migrated image

$$
\left[-\xi_{a}, \xi_{b}\right]=\left[-\xi_{a},-\xi_{c}\right)+\left(-\xi_{c}, \xi_{c}\right)+\left(\xi_{c}, \xi_{b}\right]
$$

where $\xi_{a}$ and $\xi_{b}$ are defined as the evaluation of the stationary point at the end points, and $\xi_{c}$ is defined as the stationary point inside the aperture.

$$
\begin{gathered}
-\xi_{a}=\frac{x_{g 1}}{\sqrt{4 z^{2}+x_{g 1}^{2}}} \quad \xi_{b}=\frac{x_{g 2}}{\sqrt{4 z^{2}+x_{g 2}^{2}}} \\
\xi_{c}=\frac{x_{g}}{\sqrt{4 z^{2}+x_{g}^{2}}}
\end{gathered}
$$

The migrated image is coming from the evaluation of and the location of the stationary point

$$
\begin{equation*}
I=I_{\xi_{c}}+I_{\xi_{a}}+I_{\xi_{b}} . \tag{38}
\end{equation*}
$$

Considering the two-way travel time

$$
\begin{equation*}
t=\frac{\sqrt{4 z^{2}+x_{g}^{2}}}{c} \tag{39}
\end{equation*}
$$

also $t_{g 1}$ as the evaluation of (39) at $x_{g}=x_{g 1}$, and $t_{g 2}$ at $x_{g}=x_{g 2}$.
Depending on where the aperture is located relative to the reflection point we will have, when the stationary point is inside the aperture migration

$$
\begin{align*}
A(z) & =\int d x_{g} \frac{\sqrt{32 \pi} z^{2}}{\pi^{2} c^{1 / 2} \cdot\left(4 z^{2}+x_{g}^{2}\right)^{3 / 4} \cdot\left(x_{g 2}-x_{g 1}\right)} \cdot e^{-i \frac{\pi}{4}} \int d \omega \omega^{-1 / 2} P\left(x_{g}, \omega\right) e^{-i \omega t} \\
& \left.+\int d x_{g} \frac{2 z^{3}}{\pi^{2}\left(4 z^{2}+x_{g 2}^{2}\right) \cdot\left(x_{g 2}-x_{g 1}\right)} \cdot \frac{e^{-i \frac{\pi}{2}}}{\left(x_{g}-x_{g_{2}}\right)} \int d \omega \omega^{-1} e^{-i \omega\left(t_{g 2}+\frac{x_{g 2}\left(x_{g}-x_{g 2}\right)}{c^{2} t}\right.}\right) P\left(x_{g}, \omega\right) \\
& -\int d x_{g} \frac{2 z^{3}}{\pi^{2}\left(4 z^{2}+x_{g 1}^{2}\right) \cdot\left(x_{g 2}-x_{g 1}\right)} \cdot \frac{e^{-i \frac{\pi}{2}}}{\left(x_{g}-x_{g_{1}}\right)} \int d \omega \omega^{-1} e^{-i \omega\left(t_{g 1}+\frac{x_{g 1}\left(x_{g}-x_{g 1}\right)}{c^{2} t}\right)} P\left(x_{g}, \omega\right) \tag{40}
\end{align*}
$$



Figure 3: Migrated image for bandlimited frequency and with the wavelet from the model presented in Figure 2 with a free surface at 15 m above the source and receivers considering that the stationary point is inside the aperture. As it is shown you can see the primary reflection and the free surface multiples.

The main contribution for the migrated image comes from the first term of the expression above and it is reflected in Figure 3. Figure 3 shows the migrated image of a model similar to the one in Figure 2. The migrated image contains the effect of a free surface located at $z=0$. The source and receivers are located at 15 m below the free surface. The image is calculated using equation (40), which considers the case where the stationary point is within the acquired offset region.

When the stationary point is outside the acquired offset region,

$$
\begin{align*}
A(z) & =\int d x_{g} \frac{2 z^{3}}{\pi^{2}\left(4 z^{2}+x_{g 2}^{2}\right) \cdot\left(x_{g 2}-x_{g 1}\right)} \cdot \frac{e^{-i \frac{\pi}{2}}}{\left(x_{g}-x_{g_{2}}\right)} \int d \omega \omega^{-1} e^{-i \omega\left(t_{g 2}+\frac{x_{g 2}\left(x_{g}-x_{g 2}\right)}{c^{2} t}\right)} P\left(x_{g}, \omega\right) \\
& -\int d x_{g} \frac{2 z^{3}}{\pi^{2}\left(4 z^{2}+x_{g 1}^{2}\right) \cdot\left(x_{g 2}-x_{g 1}\right)} \cdot \frac{e^{-i \frac{\pi}{2}}}{\left(x_{g}-x_{g_{1}}\right)} \int d \omega \omega^{-1} e^{-i \omega\left(t_{g 1}+\frac{x_{g 1}\left(x_{g}-x_{g 1}\right)}{c^{2} t}\right)} P\left(x_{g}, \omega\right) \tag{41}
\end{align*}
$$

The contribution to the migrated image in this case comes from the end points of the aperture as it is shown in Figure 4. $A(z)$ contains only terms calculated on the edges of the region. The migrated image corresponding to equation (41) is shown in Figure 5. Figure 5 shows


Figure 4: (a) Schematic representation of the interpolation experiment for a single interface model from pre-critical to pre-critical. (b) Shot record of the model in (a) generated by finite differences.
a degraded image compared to the one shown in Figure 4 which contains the contribution from a stationary point calculation. Since $A(z)$ deviates from zero between the higher order imaged multiples, we will expect that the data reconstruction of the higher order multiples will be worse than the data reconstruction of the primary and first order multiple.

When the stationary point is at the end point $\left(\xi_{s}=\xi_{b}\right)$,

$$
\begin{align*}
A(z) & =\int d x_{g} \frac{\sqrt{32 \pi} z^{2}}{\pi^{2} c^{1 / 2} \cdot\left(4 z^{2}+x_{g}^{2}\right)^{3 / 4} \cdot\left(x_{g 2}-x_{g 1}\right)} \cdot e^{-i \frac{\pi}{4}} \int d \omega \omega^{-1 / 2} P\left(x_{g}, \omega\right) e^{-i \omega t} \\
& \left.+\int d x_{g} \frac{2 z^{3}}{\pi^{2}\left(4 z^{2}+x_{g 1}^{2}\right) \cdot\left(x_{g 2}-x_{g 1}\right)} \cdot \frac{e^{-i \frac{\pi}{2}}}{\left(x_{g}-x_{g_{2}}\right)} \int d \omega \omega^{-1} e^{-i \omega\left(t_{g 1}+\frac{x_{g 1}\left(x_{g}-x_{g 2}\right)}{c^{2} t}\right.}\right) \tag{42}
\end{align*}\left(x_{g}, \omega\right) .
$$



Figure 5: Migrated image of data from the model shown in Figure 2 but with a free surface at 15 m above the source and receivers. This image is calculated with equation (41) using only the contribution from the end points of the offset range.

When the stationary point is at the end $\operatorname{point}\left(\xi_{s}=-\xi_{a}\right)$,

$$
\begin{align*}
A(z) & =\int d x_{g} \frac{\sqrt{32 \pi} z^{2}}{\pi^{2} c^{1 / 2} \cdot\left(4 z^{2}+x_{g}^{2}\right)^{3 / 4} \cdot\left(x_{g 2}-x_{g 1}\right)} \cdot e^{-i \frac{\pi}{4}} \int d \omega \omega^{-1 / 2} P\left(x_{g}, \omega\right) e^{-i \omega t} \\
& \left.+\int d x_{g} \frac{2 z^{3}}{\pi^{2}\left(4 z^{2}+x_{g 1}^{2}\right) \cdot\left(x_{g 2}-x_{g 1}\right)} \cdot \frac{e^{-i \frac{\pi}{2}}}{\left(x_{g}-x_{g_{1}}\right)} \int d \omega \omega^{-1} e^{-i \omega\left(t_{g 1}+\frac{x_{g 1}\left(x_{g}-x_{g 1}\right)}{c^{2} t}\right.}\right) P\left(x_{g}, \omega\right) \tag{43}
\end{align*}
$$

## 5 Extrapolation procedure

### 5.1 Data generation: the forward problem

In the previous section, we established a way to calculate an estimate of $A(z)$ based on the acquired finite aperture data. Now we have the means to predict data at each desired location on the acquisition surface. The interpolation/extrapolation procedure (Stolt, 2002) involves using the migrated image as input to the modelling formula in (13) - the forward Born approximation.

It is performed by reintroducing the inversion part $A(z)$ as a function of the data $P\left(x_{g} \mid x_{s} ; t_{s}-\right.$ $t_{g}$ ) into (13).

$$
\begin{align*}
P\left(x_{g}^{\prime} \mid x_{s}, t^{\prime}\right)= & \frac{1}{(2 \pi)^{3}} \int d k_{g x}^{\prime} \int d k_{s x}^{\prime} \int d \omega^{\prime} e^{i\left(k_{g x}^{\prime} x_{g}^{\prime}-k_{s x}^{\prime} x_{s}+\omega^{\prime} t^{\prime}\right)} \times \\
& \frac{\omega^{\prime 2}}{4 \cdot \sqrt{\frac{\omega^{\prime 2}}{c^{2}}-k_{g x}^{\prime 2}}} \frac{A\left(k_{z}^{\prime}\right)}{\sqrt{\frac{\omega^{\prime 2}}{c^{2}}-k_{s x}^{\prime 2}}} \tag{44}
\end{align*}
$$

There have been different attempts to solve (44). One approach is to extrapolate the migrated image to new locations by doing a demigration with an approximation in the output. In other words using a stationary phase approximation with respect to $k_{g x}^{\prime}$ and solving (44)

$$
\begin{align*}
& k_{x s t a t}=+\frac{x_{g}^{\prime} k_{z}^{\prime}}{2 \sqrt{t^{\prime 2} c^{2}-x_{g}^{\prime 2}}}  \tag{45}\\
& P\left(x_{g}^{\prime} \mid t^{\prime}\right)=\frac{1}{16 \pi} \frac{c^{3} t^{\prime 2}}{\sqrt{2}\left(x_{g}^{\prime 2}-c^{2} t^{\prime 2}\right)^{5 / 4}} \int d k_{z}^{\prime} e^{-i\left(\frac{\sqrt{t^{\prime 2} c^{2}-x_{g}^{\prime 2}}}{2}\right)} k_{z}^{\prime} \sqrt{k_{z}^{\prime}} A\left(k_{z}^{\prime}\right) . \tag{46}
\end{align*}
$$

This approach was presented by Stolt (2002). Using the stationary phase approximation for solving the forward problem, produces a computationally efficient scheme for data reconstruction. In this paper, we deviate from Stolt's approach by calculating the wave field. In this case, calculating the field at each location involves computing contributions from all wavenumbers, $k_{g x}$.
Changing variables from $\omega^{\prime}$ to $k_{z}^{\prime}$ and using the constraint condition (18)

$$
\begin{equation*}
P\left(x_{g}^{\prime} \mid t^{\prime}\right)=\frac{c}{8 \pi} \int d k_{g x}^{\prime} \int d k_{z}^{\prime} \frac{e^{i\left(k_{g x}^{\prime} x_{g}^{\prime}-c t^{\prime} \sqrt{\frac{k_{z}^{\prime 2}}{4}+k_{g x}^{\prime 2}}\right)}}{k_{z}^{\prime}} \sqrt{\frac{k_{z}^{\prime 2}}{4}+k_{g x}^{\prime 2}} A\left(k_{z}^{\prime}\right) . \tag{47}
\end{equation*}
$$



Figure 6: Interpolated data and the actual data at 85 m from the source when we consider an aperture starting at 70 m until 500 m with a separation of 10 m between receivers.

### 5.2 Interpolation in the pre-critical zone

To illustrate the effects of the interpolation/extrapolation schemes, we provide synthetic data examples. We consider the acoustic model defined earlier with a homogeneous half-space of water with one flat reflector at 300 m and no free surface (see Figure 4a). The velocity above the reflector is $1500 \mathrm{~m} / \mathrm{s}$ and below it is $2250 \mathrm{~m} / \mathrm{s}$.
The synthetic shot gather is generated using a finite difference modelling algorithm and using a Ricker wavelet with a center frequency of 30 Hz . The data are sampled at 1 ms , and the record length is two seconds. The shot gather is displayed in Figure 4b.

The algorithm is evaluated on a shot gather where the offset ranges from $70-500 \mathrm{~m}$. The receiver interval is 10 m . We interpolate a trace at an offset of 85 m . Figure 6 shows the results and comparison between the actual modelled data at 85 m and the reconstructed data. The two traces are in good agreement. The interpolated trace appears to be a bit broader than the reference trace.

### 5.3 Extrapolation from post-critical to pre-critical

We consider a shot gather containing measurements in the offset range $900-1000 \mathrm{~m}$ where the receiver spacing is 5 m . The geometry is shown in Figure 7. As shown, the data are measured in the post-critical regime. Our goal is to use the post-critical data for doing inversion and then extrapolate data to the pre-critical zone. An interpolated trace is calculated at offset 475 m and compared with the finite difference modelled trace in Figure 8. There is a considerable phase shift between the two traces. The reason is that the reflection coefficient of the post-critical data is phase-shifted with respect to the pre-critical reflection coefficient. This affects our interpolation results since $A(z)$ is calculated from the post-critical data and will therefore include that phase shift. That phase shift is "propagated" into the new trace when doing the extrapolation.
Several tests have been performed with different apertures beyond the critical angle. The presence of the head wave has an effect on the amplitude. From post-critical to pre-critical data it seems that the reconstructed amplitude increases the accuracy when the presence of the head wave does not mitigate the original signal, as it is shown in Figures 9 and 10. As long as you interpolate or extrapolate to certain limits the procedure is successful. According to this the cascade process of inverse and forward is constrained to the first step.

## 6 Data reconstruction for data with free surface multiples

As discussed in the introduction, accurate data reconstruction is a practical prerequisite for the most effective multiple attenuation algorithms and must therefore be able to interpolate/extrapolate multiples as well as primaries.


Figure 7: Schematic representation of the extrapolation experiment for a single interface model from post critical to pre-critical.

We consider the model described in Figure 11 with a free surface at $z=0 \mathrm{~m}$. The source and receivers are located 15 m below the free surface. A finite difference scheme is used to generate the synthetic data and the shot gather is displayed in Figure 11 with the primary at around 0.4 sec . All other events are free surface multiples.
Our goal is to reconstruct data in the near offset region from data acquired in the offset range $150-500 \mathrm{~m}$. The receiver spacing is 5 m . Figure 12 presents the comparison of reconstructed and original data at 100 m from the source. The first event is a primary reflection, which is followed by the first and second order free surface multiples. We note that both the timing and amplitude of the reconstructed data are comparable with the original data. For the objective of multiple prediction, the most important issue is the one of predicted time, and this data reconstruction scheme has calculated the time precisely.
The reconstruction of the primary reflection versus the original one is shown in Figure 13. The amplitude is in the same order of magnitude as the original data, while the time is precisely estimated. The reconstruction of the first order multiple and its original one is shown in Figure 14. Under certain considerations you can always extrapolate data with free


Figure 8: Reconstructed and original data at 475 m from the source from a collection of data between offsets 900-1000 m in the post-critical zone and a separation of 5 m .
surface multiples to new locations. The second order multiple it is also shown in Figure 15 , there is no time delay between the reconstructed and the original one only under the consideration of extrapolating from pre-critical to pre-critical.

It appears that our algorithm properly reconstructs both primaries and multiples which is a necessary requirement for our objectives in multiple attenuation.

## 7 Contributions of the end points to the data reconstruction

If the aperture is compensated using a neutralizer function, there is an effect of the end points in the data reconstruction. As was shown previously, the migrated image (see Figure


Figure 9: Reconstructed and original data at 475 m from the source from a collection of data between 9001000 m in the post-critical zone and a separation of 5 m shifted in time to make a comparison with the original data.
4) is divided into three parts (37). The main contribution comes from the stationary point inside the aperture, followed by two terms coming from the end points.

To see how the end points contribute to the reconstruction of the data, we have reconstructed the data with and without contributions from the end points using the data with multiples. Figure 16 shows the reconstructed primary with and without the end point contribution. There is almost a perfect fit between the two traces which indicates that the end points is small in this specific example.

Figure 17 displays the reconstructed data for the first order multiple. As in Figure 16, we see no significant difference between the data reconstructed with and without the end point contribution. The comparison between the original data and the reconstructed with the contributions of the end points shows no improvement in amplitude.


Figure 10: Difference between the reconstructed data and actual data at 475 m from Figure 9. Notice the change in amplitude due to the presence of the head wave.

## 8 Seismic data reconstruction for free surface multiple removal

There is recent interest in multiple attenuation technology resulting from current exploration challenges, e.g, in deep water with variable water bottom, in subsalt or shallow water exploration (Carvalho, 1992; Weglein et al., 2003).

These cases are representative of circumstances where 1-D assumptions are violated and subsurface information is not available. Inverse scattering multiple attenuation is designed to address these problems. It requires near source traces and other prerequisites. Motivated by this, we tested seismic data reconstruction with an eye to extrapolating traces for the inverse-scattering series multiple attenuation method.

The free surface demultiple algorithm (Weglein et al., 1997) can be summarized as follows:


Figure 11: (a) Schematic representation of the extrapolation experiment for a single interface model from pre-critical to pre-critical for a single interface with two medium velocities $c=1500 \mathrm{~m} / \mathrm{s}$ and $v=2250 \mathrm{~m} / \mathrm{s}$ and with a free surface at 15 m above the source and receivers. (b) Shot record of model (a) generated with finite differences.

- The Data, $D$, are computed by subtracting the reference field, $G_{0}=G_{0}^{d}+G_{0}^{F S}$ from the total field, $G$, on the measurement surface.
- Compute the deghosted data, $\tilde{D}$, where $\tilde{D}=D /\left[\left(e^{2 i q_{g} \epsilon_{g}}-1\right)\left(e^{2 i q_{s} \epsilon_{s}}-1\right)\right], \epsilon_{g}, \epsilon_{s}$ are source and receiver depths, and $q_{g}, q_{s}$ are vertical wave numbers.
- For a 1D Earth $\tilde{D}=2 \pi \delta\left(k_{g}-k_{s}\right) \cdot \tilde{D}(k, \omega)$
- The series of deghosted and free-surface demultiple data, $\tilde{D}$, is given

$$
\begin{equation*}
D^{\prime}(k, \omega)=\sum_{n=1}^{\infty} \tilde{D}_{n}(k, \omega) \tag{48}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{D}_{n}(k, \omega)=\frac{2 q}{i \rho_{r} B(\omega)} \cdot e^{i q\left(\epsilon_{g}+\epsilon_{s}\right)} \cdot \tilde{D}_{1}(k, \omega) \tilde{D}_{n-1}(k, \omega) . \tag{49}
\end{equation*}
$$

We are going to generate one shot gather for a 1D Earth with a reflectivity code, see Figure 18, with a point source and receiver at a distance 30 m from the free surface.

We have a two layer model with $1500 \mathrm{~m} / \mathrm{s}$ velocity in the first layer and $1800 \mathrm{~m} / \mathrm{s}$ in the second layer. The minimum frequency is 10 Hz and the maximum is 80 Hz . The time sample rate is $d t=0.001 \mathrm{~s}$, and the offset rate is $d x=2 \mathrm{~m}$. We have taken into account a $Q$ absorption value of 200 and constant density.

The first order free surface multiple can be seen in Figure 19. The next task is to evaluate seismic data reconstruction for a shot record with missing traces. Under this consideration we have used the original data from Figure 18 and eliminated some traces along the shot.

Using interpolation and extrapolation considerations we have been able to reconstruct the original data as it is shown in Figure 20b. Finally, this reconstructed data has been introduced as an input in the free surface demultiple algorithm. Figure ?? shows the results after removing multiples from this reconstructed data.

## 9 Conclusions

We have presented a procedure to interpolate and extrapolate traces to new locations from positions recorded. It consists of an integral operator considering the data in a continuous medium and is based on the assumptions that principle of superposition, physical invariance are satisfied. It also assume constant reference velocity and high frequency approximations.

The first test points out the crucial implication of what $V_{1}$ means. You can successfully interpolate traces and extrapolate from pre-critical to pre-critical and from post-critical to post-critical. The failure to extrapolate from post-critical to pre-critical in shows the intrinsic relation between the data and $V_{1}$. As mentioned earlier, the expression (9) is exact and precise without a thought or care about convergence, and it is the heart of the extrapolation scheme.

The first task, the inversion is performed via a stationary phase approximation. Under this consideration we have studied the implications to finite data, and we have calculated an aperture compensating factor. It was also shown that depending on the use of a neutralizer function or a taper function, the approximation breaks down because the integrand is not smooth. The contributions from the end points of the aperture need to be calculated to give validity to the stationary phase approximation.

The second task, demigration, has been accomplished using full wave theory, avoiding the same approximation that was made at the outset. We can summarize our results as follows:

- The reconstruction of seismic data can be done effectively in the pre-critical region.
- Extrapolating from post-critical to pre-critical confronts us with phase problems. There is a phase shift problem due to the reflectivity in the post-critical region.
- Corrected terms to the stationary phase approximation when we use a neutralizer function as an aperture compensator were calculated, but it doesn't improve the reconstructed amplitude.
- Seismic data reconstruction with free surface multiples for extrapolation near the source can be generated. Not only the primary reflection but also the first and second free surface multiples were estimated at the correct time using this extrapolation.
- Reconstructed data was presented from original data with missing traces, and the result of the reconstructed traces were shown.
- With the successful result of predicting the right time for the free surface multiple and the primary reflection we show the effect of the free surface removal on this reconstructed data.


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## A Stationary phase approximation

Study of the contributions to the stationary phase approximation from the critical points depending on their location. Stationary point $\xi_{s}$ near the end point $-\xi_{a}$ :

$$
\begin{gather*}
I=I_{s}-I_{\xi_{a}}-I_{\xi_{b}}  \tag{50}\\
I_{\xi_{a}}=H(-\epsilon) \cdot I_{s}+\epsilon \sqrt{\frac{2}{\left|\omega \tau^{\prime \prime}\left(-\xi_{a}\right)\right|}} \times F_{\gamma}(u) G\left(-\xi_{a}\right) e^{i\left(\gamma u^{2}\right)}  \tag{51}\\
\gamma=\operatorname{sgn}\left(\omega \tau^{\prime \prime}\left(-\xi_{a}\right)\right) \quad \epsilon=\operatorname{sgn}\left(\xi_{a}+\xi_{s}\right) \quad u=\left|\tau^{\prime}\left(-\xi_{a}\right)\right| \sqrt{\frac{|\omega|}{\left|2 \tau^{\prime \prime}\left(-\xi_{a}\right)\right|}}  \tag{52}\\
\sigma=\operatorname{sgn}\left(\omega \tau^{\prime \prime}\left(\xi_{s}\right)\right) \quad G(\xi)=E(\xi) e^{i \omega \tau\left(\xi_{s}\right)} \quad F_{\sigma}(\eta) \approx \frac{e^{i \sigma\left(\eta^{2}+\pi / 2\right)}}{2 \eta} \\
I_{\xi_{b}}=-\frac{1}{i \omega} \frac{G\left(\xi_{b}\right)}{\tau^{\prime}\left(\xi_{b}\right)} \tag{53}
\end{gather*}
$$

$$
\begin{equation*}
I_{s}=\frac{\sqrt{2 \pi}}{\left|\omega \tau^{\prime \prime}\left(\xi_{s}\right)\right|^{1 / 2}} G\left(\xi_{s}\right) e^{i \sigma \frac{\pi}{4}} \tag{54}
\end{equation*}
$$

In the same way if we have the stationary point near the endpoint $\xi_{s}$ near $\xi_{b}$ :

$$
\begin{gather*}
I=I_{s}-I_{\xi_{a}}-I_{\xi_{b}}  \tag{55}\\
I_{\xi_{b}}=H(-\epsilon) \cdot I_{s}+\epsilon \sqrt{\frac{2}{\left|\omega \tau^{\prime \prime}\left(\xi_{b}\right)\right|}} \times F_{\gamma}(u) G\left(\xi_{b}\right) e^{i\left(\gamma u^{2}\right)}  \tag{56}\\
I_{\xi_{a}}=-\frac{1}{i \omega} \frac{G\left(\xi_{a}\right)}{\tau^{\prime}\left(\xi_{a}\right)}  \tag{57}\\
I_{s}=\frac{\sqrt{2 \pi}}{\left|\omega \tau^{\prime \prime}\left(\xi_{s}\right)\right|^{1 / 2}} G\left(\xi_{s}\right) e^{i \sigma \frac{\pi}{4}} \tag{58}
\end{gather*}
$$

In the situation of having the stationary point at the endpoint: $\xi_{s}=-\xi_{a} \Rightarrow I=\frac{I_{s}}{2}-I_{\xi_{b}}$ or in the case $\xi_{s}=\xi_{b} \Rightarrow I=\frac{I_{s}}{2}-I_{\xi_{a}}$.

When the stationary point is outside the migration aperture:

$$
\begin{equation*}
I=\frac{1}{i \omega}\left[\frac{G\left(\xi_{b}\right)}{\tau^{\prime}\left(\xi_{b}\right)}-\frac{G\left(-\xi_{a}\right)}{\tau^{\prime}\left(-\xi_{a}\right)}\right] \tag{59}
\end{equation*}
$$



Figure 12: Reconstructed and original data from the model in Figure 2 with the source and receivers 15 m below free surface extrapolated at 100 m from an offset range $150-500 \mathrm{~m}$ and a distance between receivers of 5 m .


Figure 13: Zoom of the primary reflection from Figure 12 where the original and reconstructed data are shown.


Figure 14: Zoom of the first order free surface multiple from Figure 12 where the original and reconstructed data are shown.


Figure 15: Zoom of the second order free surface multiple from Figure 12 where the original and reconstructed data are shown.


Figure 16: Comparison of the primary reflection reconstructed with and without the end point contributions. As it is shown the contributions of the end points are small for this specific example. In general for dipping reflectors the influence can be estimated to be high.


Figure 17: Comparison of the first order free surface multiple reconstructed with and without the end point contributions. As it is shown the contributions of the end points are small for this specific example.


Figure 18: Shot record gather with the source and receivers situated 30 m below free surface.


Figure 19: Inverse polarity of Figure 18 to show the first order multiple we want to remove.


Figure 20: Comparison of the original data with missing traces (a) and the reconstructed data (b).


Figure 21: Preliminary comparison of (b) the data from Figure 20 after data reconstruction with (a) the result of the free surface multiple prediction using the reconstructed data (displayed with inverse polarity to highlight the first order free surface multiple). The timing of the multiples has been well-predicted with the multiple prediction algorithm. The low amplitude on the near offsets of the prediction is due to the fact that the input data contained only positive offsets and no attempt was made to synthesize split-spread acquisition.

# MULTIPLE ATTENUATION 

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Geophysics Reprint Series

## Preface

Papers selected for this SEG reprint volume on multiple attenuation sample the geophysical literature from 1948 through 2003. For the past fifty-odd years the presence of multiply reflected energy in seismic data has been a serious issue and challenge for geophysicists. It remains so today. Nevertheless, the seismic exploration community has made dramatic progress in multiple attenuation. This volume chronicles and examines the amazing history and evolution of methods for attenuating multiples. The papers are organized into nine thematic chapters, and appear chronologically (sometimes within subtopics) within each.

The volume begins with a short introduction to some basic concepts of multiple attenuation. Next, Chapter 1 focuses on the era when multiple reflections in seismic data were first clearly seen and characterized. Interestingly, prior to the late 1940's many geophysicists thought that multiples were so weak that they would never be seen at all, much less be a significant problem. Chapter 2 describes some of the first efforts at attenuating multiples - signal processing algorithms that deconvolve or otherwise remove periodic events in seismic traces - as well as some more recent developments using such approaches. In contrast, the papers in Chapter 3 describe methods that attenuate multiples based on moveout discrimination or some other event characteristic that distinguishes them from primary reflections. Papers describing the popular Radon transform-based method of multiple attenuation appear in this chapter.

Chapters 4 and 5 describe two categories of multiple attenuation methods that are directed at accommodating a fully multi-dimensional subsurface: (1) linear methods of modeling and subtracting multiples that require an explicit or implicit model of the reflections that generate multiples; and (2) nonlinear methods that do not require such a model. The latter of these represents a major conceptual advance: the idea that, using nonlinear multidimensional wave equation-based methods, multiple reflections can be fully predicted from the information contained within a seismic data set independently of any assumptions or knowledge about the subsurface. Once so predicted, multiples can be subtracted from the original data, yielding a multiple-free result. Because practical applications based on this idea are relatively new, many geophysicists who read this volume may not have been exposed to it during their formal education. Hence, following our Chapter 4 introduction we have included a brief tutorial section that explains the basic concepts of this type of multiple prediction.

There are two complementary ways of thinking about nonlinear wave equation-based multiple prediction. Chapter 4 covers the free-surface and interface model, and Chapter 5 the freesurface and point scatterer model. Compared to earlier methods of multiple attenuation, these multiple prediction schemes place certain conditions on the seismic experiment rather than requiring assumptions about the nature of the subsurface. For example, the source signature must be known, and, as in migration, the data set aperture becomes important. The papers in Chapter 6 discuss some of the consequential difficulties and issues that must be dealt with to make nonlinear wave equation-based multiple prediction practical for field data sets. In particular, many of the Chapter 6 papers describe ways of coping with the 3-D
nature of the primary and multiple wavefields when the acquisition experiment itself is not spatially sampled in a full 3-D sense.

The words "multiple attenuation" immediately make many geophysicists think of marine streamer data. Chapter 7 reminds us that multiples are also an important problem for land data sets and marine data recorded by methods other than horizontally towed streamers. Chapter 8 contains a collection of tutorial papers that emphasize recent multiple attenuation concepts and methodologies. Finally, Chapter 9 presents an alternative approach to dealing with multiple reflections - using them as signal to enhance the subsurface image rather than considering them as noise that must be removed. The serious challenges facing such an endeavor are also described.

Editors of a reprint volume usually face two dilemmas: what criteria are used to select the papers and how many papers should be included. Paper selection was difficult - and necessary - because no reprint volume could contain all the worthy papers on a subject as broad as multiple attenuation. Except for Chapters 1 and 9, we decided to include only papers that directly address the attenuation of multiples. This basic criterion meant that papers about technology that could be used for multiple attenuation, but that contained little or no discussion of multiple attenuation itself, were not selected. Thus, for example, no papers on the fundamentals of the Radon transform or predictive deconvolution appear in this volume. Having established the basic criterion, we then selected papers that we considered to be important and influential. We also polled some of our colleagues and acquaintances for suggestions, especially for early papers about multiple attenuation. Finally, we scanned through the references of selected papers, searching for additional papers that those authors had considered important.

Our objective in this volume is to collect, synthesize, and provide a perspective for the literature on multiple attenuation, an important cornerstone of seismic data processing. Not all of the papers in this volume received a traditional peer review prior to their original publication. In recent years papers in The Leading Edge, First Break, and the SEG and EAGE Expanded Abstracts have formed an increasingly important repository of technical literature. We feel - and we hope that the readers will agree - that including papers from those sources falls within our objective.

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## Art Weglein and Bill Dragoset

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## Volume Introduction

Most of the papers in this reprint volume address the problem of multiple reflections that appear in marine seismic data. In seeking to analyze such data, it helps to classify the recorded events according to the reflections and propagations they have experienced. For typical towed streamer exploration, both the seismic source and the receivers are deployed within the water layer. Figures 1, 2, and 3 show examples of various classes of events for streamer data. The first classification separates events that have experienced the earth's subsurface below the water layer from those that have not. The latter category (Figure 2a) consists of two events:

- the direct arrival, which is energy that travels in a straight-line path from source to receiver, and
- the direct arrival ghost, which is energy that propagates upwards from the source, reflects off the water surface, and then straight to the receiver.

All other recorded events either propagate within the earth or at least encounter the earth at the water bottom. (Seismic energy can reflect from sharp velocity or density contrasts within the water layer. This happens only rarely, so such events are not considered in the classification scheme described here.)

The category of events that experience the earth includes reflections, refractions, diffractions, mode conversions, and an infinite variety of combinations thereof. These events can be further subdivided into two main categories:

- events that propagate downwards from the source and are recorded as up-going waves at a receiver, and
- events that either propagate upwards from the source and/or are recorded as down-going waves at a receiver.

These latter events are called ghosts, of either source or receiver variety, respectively (Figure 2).

Excluding the ghosts, the remaining reflection events can be classified as either primary or multiple depending on the number of upward reflections experienced:

- primary events experience one upward reflection (Figure 1a), and
- multiple events experience two or more upward reflections (Figures 1b-1f).

Here, an upward reflection is defined as one where the incident wave moves away from the measurement surface (that is, the streamer) towards the reflector and the reflected wave moves away from the reflector towards the measurement surface. The incident wave in a downward reflection moves towards the measurement surface and then away from the measurement surface after reflection. Reflections are possible that are neither upward nor downward by these definitions. An example is shown at the right-hand side of Figure 3a. Nevertheless, traditionally, primaries and multiples are defined only in terms of the number and location of upward and downward reflections in their raypaths. The event in Figure 3a has one upward reflection, no downward reflections, and one reflection that is neither; hence, it is a primary.

The final classification defines particular types of multiply reflected events based on the location of their downward reflections:

- A multiple with one or more downward reflections at the free surface (that is, the water surface) is called a free-surface multiple, independent of the rest of its trajectory.
- An internal multiple has all of its downward reflections below the free surface.

Interbed multiple is a common alternative name for internal multiple. Note that although source and receiver ghosts are free surface-related events that experience a downward reflection at the free surface, they do not qualify as free-surface multiples because they were excluded at an earlier step in the classification scheme.

Sound waves in the earth sometimes experience more complicated raypaths than alternating up and down reflections. For example, events that do not easily fit into the classes described above include those that experience refractions or diffractions within their history. The event in Figure 3b has only one upward reflection, but that event is clearly not a primary. The need to accommodate a broader range of events, coupled with advances in multiple attenuation theories and algorithms, suggests the need for more general definitions:

- A prime event (or primary) is a recorded event that cannot be decomposed into other events recorded within the same data set.
- A composite event (or multiple) is a recorded event that can be decomposed into a number of other events that appear within the same data set.

The phrases "can be decomposed" and "cannot be decomposed" refer to whether or not the travel time of the event in question can be expressed as sums and differences of the travel times of other events within the data set (see Figure 3). The concept of "data set" refers to a complete set of shot records that spans sufficient spatial aperture to record all of a multiple's composite events. Note that the decomposition of higher-order multiples can be accomplished in more than one way, and can include events that are themselves multiples. For example, the multiple in Figure 3 f can be decomposed into primary events 1-3, 2-3, 2-4, and $4-5$ or alternatively into a multiple event, $1-4$, and a primary event, $4-5$. This more general scheme of classifying primary and multiple events becomes important in Chapters 4 and 5 , which include comprehensive methods that attenuate all events that are composites within the measurement set, including event types beyond the simple upward/downward reflection definition. Further details are presented in the Chapter 4 tutorial.

In the literature, one finds mention of multiple attenuation, elimination, and suppression. Generally, these terms are used interchangeably. In this volume, however, we assign specific definitions to these terms. Attenuation and suppression are synonyms that refer to a process in which the amplitudes of the multiple events in a seismic data set are reduced. Elimination refers to a process in which at least one class of multiples is completely removed from a seismic data set. Thus, elimination is a form of attenuation in which the amplitude reduction is complete. This distinction is important. Some multiple removal algorithms are, at least in principle, eliminators, while others are only attenuators. There is also an important and increasingly significant difference between an explicit prediction of amplitude and phase (time) of a multiple and the assumption that whatever falls on a given traveltime
trajectory is a multiple to be eliminated. In practice even an elimination algorithm usually accomplishes only partial multiple removal because field data sets seldom meet all of the prerequisites for elimination. This is an important issue, but it is unrelated to the intrinsic capability of the algorithm. The distinction between the intrinsic capabilities of a method and limitations imposed on those capabilities by external factors (such as data collection, subsurface assumptions, etc.) is important for two reasons: an understanding of when application of a procedure is appropriate, and clarity in attributing lack of effectiveness to the proper cause, thereby guiding those seeking better results in the future.

In geophysics, as in all fields of science, progress sometimes appears to be somewhat chaotic. Over a period of many years, however, an evolutionary pattern often emerges from the chaos. Table 1 shows such a pattern for the discipline of multiple attenuation. The leftmost column lists advances in the evolution of multiple attenuation that are presented in Chapters 2 through 5 of this volume. The next column indicates the complexity and realism of the physical models that the attenuation methods can accommodate. More checks indicate higher complexity, realism, and completeness of the physics behind the method. For example, algorithms that attenuate multiples based on periodicity are based on a simple 1-D model of the subsurface, whereas moveout discrimination methods are based on a more complex 2-D ray-tracing model. Together, these two columns represent an overall trend: more complete, realistic physics allows significant advances, and hence improvements, in multiple attenuation. Such improvements reduce the risk of producing poor quality processed seismic data sets, and thus, ultimately, lower the overall risk of hydrocarbon exploration itself. The third column ranks the complexity of the ancillary subsurface information needed by each multiple attenuation method. For example, moveout discrimination requires a nominal velocity profile, and, if one wishes to attenuate selected interbed multiples, the free-surface and interface wavefield method requires specification of a multiple-generating horizon or region. The fourth column describes the need for description of the seismic experiment, such as information about the source signature and the receiver depths. Finally, the rightmost column indicates the data acquisition and computational burden of each method.

Table 1 shows that as the complexity of the physical model that a method can accommodate increases, the burden on data acquisition and processing likewise increases. Simply accommodating a higher dimension of variability in the subsurface requires more thorough surface data acquisition, regardless of the nature of the seismic processing algorithm. Consider the bottom row in the table. Applying a true 3-D version of the free-surface and point scatterer method to attenuate all multiples requires not only a massive computational effort, but also wide-aperture, full-azimuth recorded data. A skeptic, then, might be inclined to ask, "Why develop such an algorithm?" The short answer to this question can be found two rows up the table. When development of the 2-D free-surface and interface model began in earnest during the 1980's, the skeptic, justifiably, could have asked the same question. Yet today, many years later, that technology is being used routinely to process data from large marine streamer 3-D surveys. This example teaches an important lesson for everyone in the exploration geophysics business: If a method is discovered that solves an important problem, eventually computational technology and data acquisition practices will evolve to
a point that makes that method practical. This is the nature of progress in the science of seismic exploration.

An interesting, and perhaps counterintuitive, aspect of recent advances in multiple attenuation is that the complexity of the required ancillary subsurface information and, more broadly, the necessary assumptions about the earth's subsurface do not increase as the attenuation algorithms reach their ultimate sophistication. Instead, these new algorithms shift the responsibility for providing detailed subsurface information upwards to the data acquisition at the surface. That shift has an important practical implication - success at removing multiples is no longer limited by uncertainty in subsurface assumptions and information, but instead by the effort one is willing to expend on data acquisition and processing. Prior to these new developments, success in removing multiples was not necessarily commensurate with money spent. With the new technology, however, an Exploration and Production company interested in paying more to achieve better multiple attenuation has that option.

Table 1 could have had many more columns and rows. For example, we could have listed the advantages and disadvantages of each method, their success or failure at attenuating different types of multiples, the years during which the methods were first proposed and then widely practiced, and so on. However, doing that would have spoiled some of the fun readers of this volume will have discovering or rediscovering the history of multiple attenuation.

| Multiple attenua- <br> tion method | Subsurface com- <br> plexity and reality <br> that method can <br> accomodate | Requirement for <br> ancillary subsur- <br> face information <br> and interpretive <br> intervention | Requirement for <br> the description <br> of seismic experi- <br> ment (e.g., source <br> signature) | Requirements on <br> data acquisition, <br> reconstruction, <br> regularization and <br> computation |
| :--- | :--- | :--- | :--- | :--- |
| Deconvolution <br> based on periodic- <br> ity of multiples | $*$ | $*$ | $*$ |  |
| Moveout discrimi- <br> nation methods | $* *$ | $*$ | $*$ |  |
| Free-surface and in- <br> terface model, 2-D <br> and pseudo 3-D | $* * *$ | $* *$ | $*$ | $* *$ |
| Free-surface and <br> point scatterer <br> model, 2-D and <br> pseudo 3D | $* * * *$ | $*$ | $* *$ | $* * *$ |
| Free-surface and <br> interface and free- <br> surface and point <br> scatterer models, <br> true 3-D | $* * * *$ | $* * *$ | $* * *$ |  |

Table 1: Major stages in the evolution of multiple attenuation.


Figure 1: Traditional definition of primary and multiple events. The blue-green area represents the water layer. The red and yellow dots indicate the positions of seismic sources and receivers, respectively. The white lines are raypaths of the events being defined. For the sake of simplicity in the figure, the rays do not refract as they cross reflecting horizons.
a) A primary event has one upward reflection.
b) A multiple event has at least two upward reflections. This example is a $1^{\text {st }}$-order free-surface multiple because it has a single downward reflection generated by the water surface.
c) $A \mathscr{Z}^{\text {nd }}$-order surface multiple.
d) This event is a $1^{\text {st }}$-order internal multiple because the generating horizon that produces the downward reflection is located in the subsurface.
e) $A 2^{\text {nd }}$-order internal multiple.
f) Hybrid event 1-3 is classified as a free-surface multiple. Even though one generating horizon is below the surface, an algorithm that attacks free-surface multiples will attenuate this event. However, event 1-2, an internal multiple, will remain in the data set after surface multiple attenuation.


Figure 2: Definition of direct arrivals and ghosts. The blue-green area represents the water layer. The red and yellow dots indicate the positions of seismic sources and receivers, respectively. The white and green lines are raypaths of the events being defined. For the sake of simplicity in the figure, the rays do not refract as they cross reflecting horizons.
a) A direct arrival (white) and its ghost (green).
b) A source ghost.
c) A receiver ghost.
d) A primary reflection with both a source ghost and a receiver ghost.
e) $1^{\text {st }}$-order surface multiple reflection with both ghosts.
f) $\mathscr{2}^{\text {nd }}$-order multiple reflection with both ghosts. Usually, removal of direct arrival and ghost events during seismic data processing is a separate issue from attenuating multiples.


Figure 3: A more general definition of a multiple is: a recorded event that can be decomposed into a number of other recorded events. The dashed green lines in e) and f) represent events whose travel times are subtracted to obtain the multiple's travel time (see below).
a)This event is classified as a primary because it cannot be decomposed into other events.
b) In the traditional classification this event is a primary because it has only one upward reflection. The more general definition classifies it as a multiple since it can be divided into two other recorded events, event 1-2 and event 2-3.
c) This event is a primary reflection. It cannot be decomposed into other recorded events.
d) This event is a multiple because it can be divided into two other recorded events: the primary reflections 1-2 and 2-3. The travel time of the multiple is the sum of the travel times of the two primaries.
e) An internal multiple composed of events 1-3, 2-3, and 2-4. The multiple's travel time is the sum of the travel times of events 1-3 and 2-4, minus the travel time of event 2-3.
f) Another multiple. The sum of travel times 1-3, 2-4, and 4-5, minus travel time 2-3 gives the multiple's travel time.

## Chapter 1

## Historical papers: characteristics of multiples

From today's perspective the January 1948 issue of Geophysics makes for fascinating reading. The topic of that issue was multiple reflections - not so much about how to attenuate them, but rather about whether they even existed in seismograms recorded on land. Consider, for example, the very first sentence in the first paper of this volume: "On several occasions ... various geophysicists have expressed ... doubt regarding the existence of multiple reflections ..." (Dix, 1948). Dix went on to argue in favor of the existence of multiples, but in the end he conceded "The evidence ... does not yet make their existence 'strictly certain' ..." Hansen (1948), however, had no doubts about the existence of multiples. He reports identifying them by their low average velocities in reflection velocity profiles from Argentina. Interestingly, the translator of Hansen's paper from its original Spanish had to remove the paper's "slightly defensive tone," which he attributed to the fact that ". . . when Mr. Hansen's paper was originally published ...few geophysicists and fewer executives reacted favorably to the idea that multiple reflections could be of any practical importance in seismograph exploration ..."

Many of the papers in the January 1948 issue of Geophysics were presented orally two years earlier at a symposium during the SEG's 1946 meeting in Los Angeles. The lead paper of that symposium (Ellsworth, 1948) presents solid evidence of multiple reflections in seismic records from Sacramento Valley, California. Ellsworth also describes several kinds of multiples, naming them Type 1, Type 2, and Type 3. Today geophysicists call those types surface multiples, peg-leg multiples, and near-surface multiples. Johnson (1948) presents indisputable evidence from Butte County, California of multiple reflections between a basalt layer and the bottom of the weathering zone. Even those geophysicists of the 1948 era who believed that multiples were present apparently were not too concerned. For example, Ellsworth concluded his paper with the statement "...the multiple-reflection question as a whole does not seem to present a serious limitation to seismograph interpretation except in isolated cases." One exception to that widely held viewpoint was Johnson's, who perhaps had a glimmer of the future when he wrote in his paper's conclusion "Thus a highly suspicious attitude toward every reflection in areas known to return some multiple reflections seems to be justified."

In marine seismograms geophysicists often observed a mysterious phenomenon dubbed "singing." As reported by Werth et al. (1959) and later by Levin (1962), singing marine seismograms were dominated by sinusoidal, nearly constant frequency energy. Singing was clearly a localized phenomenon, often appearing and then disappearing several times along the length of a single seismic line. Werth et al. describe the recording and analysis of an experiment designed to reveal the cause of singing. They concluded that, at least in their test area, singing was caused by short-period multiple reverberations in the water layer rather than by a wave guide-like excitation of the water layer by the seismic source. Levin's paper describes an experiment to understand the seismic properties of Lake Maracaibo,
which even today has a reputation for producing seismic records that are difficult to process and interpret. Levin found that singing was associated with areas having a high waterbottom reflection coefficient caused by low velocity in gas-saturated bottom mud, a situation certainly conducive to multiple generation.

This set of papers provides a history of significant pioneering and discovery. By their example, they serve to guide and encourage those striving for new scientific understanding.

## Chapter 1 Papers

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## Chapter 2

## Multiple attenuation based on a convolutional model

Multiple reflections have many properties that can distinguish them from primaries. The papers in this chapter discuss algorithms that, in essence, attenuate multiples by exploiting one of those properties, their periodicity. Multiply reflected energy is truly periodic only in a 1-D medium. Nevertheless, short-period multiples, such as those that occur in many shallow-water layer areas, are nearly periodic, making simple inverse filtering or deconvolution viable methods of eliminating such multiples. For example, in his classic paper about water reverberations Backus (1959) treats the effect of the water layer as an approximate linear filter, and subsequently removes the water-layer reverberations by convolving seismic traces with the inverse of that filter. Goupillaud (1961) presents a non-linear generalization of the inverse filter concept, which ideally removes the effect of reverberations in a shallow layer for either land or marine seismic data.
Watson (1965) proposes a simple 1-D scheme that models first-order surface multiples as the convolution of a reflectivity sequence with itself (with an adjustment for the surface reflection coefficient). This leads to an inverse equation for a multiple-free primary trace. Watson approximately solved that equation with a feedback loop procedure. Anstey and Newman (1966) discuss the auto-correlogram, which measures the periodicity of seismic traces, and the retro-correlogram (Watson's method under another name), which predicts multiples. They use the auto-correlogram to determine the presence of multiples and suggest, like Watson, the use of a feedback mechanism with the retro-correlogram to attenuate multiples. Many of the papers in Chapter 4 (e.g., Berkhout and Verschuur, 1997) discuss a multiple prediction method that is essentially a 2-D generalization of the idea in the Watson and Anstey and Newman papers. The differences are interesting. In Watson's equation (9), for example, wavelet term $c_{1}(t)$ "approximates the additional filtering provided the multiples by their relatively longer paths in the more highly attenuating near-surface formations." In a 2-D prediction, such a term is not required because the prediction operator accounts for such effects automatically. Anstey and Newman recognized that their 1-D retro-correlogram did not produce accurate results for large offsets or dipping events. A 2-D prediction, on the other hand, does incorporate the effects of offset and dip. The paper by Kunetz and Fourmann (1968) offers two efficient schemes for computing 1-D-based multiple deconvolution operators.
The final three papers in this chapter extend the deconvolution approach to multiple attenuation to situations where the multiple reflections are not periodic. Taner et al. (1995) accomplish this by multichannel predictive deconvolution performed in the $x-t$ domain. This paper is notable for its thorough introduction to the concept of multichannel deconvolution and how that approach is related to other multiple attenuation methods. Lamont and Uren (1997) introduce a "multiple moveout" procedure that makes multiples periodic, and follow that by "isostretch radial trace" to stabilize the wavelet in time. Together, the two transformations pre-condition multiple events to make them suitable for a simple 1-D deconvolution.

Parrish (1998) describes regularizing water-layer multiples spatially and temporarily by migrating the data using the water velocity. This makes single trace dereverberation more effective. After multiple attenuation residual migration relocates the remaining primary reflections to their final positions.

Overall, the papers in this chapter show that the convolutional model has provided tremendous practical value. Simultaneously, it planted the seeds that grew into multi-dimensional wave theoretic methods for dealing with multiple reflections.

## Chapter 2 Papers

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## Chapter 3

## Multiple attenuation based on event characteristics

Aside from periodicity or near periodicity (see Chapter 2), multiple reflections can differ from primary reflections in other ways that can be exploited to attenuate them. Chief among these is differential moveout. For some seismic data sets simple CMP stacking is effective at reducing the amplitudes of multiples. In the words of Harry Mayne, inventor of the CMP method, "reflections which follow the assumed travel paths are greatly enhanced, and other events are reduced" (Mayne, 1962). When stacking alone is insufficient, better separation of primary and multiple events can be achieved by weighted stacking. Schneider et al. (1965) demonstrate a method of weighted stacking where the weights are determined by prestack application of optimal filters designed by a multichannel least-squares method. Schoenberger (1996) presents an excellent tutorial on the subject of weighted stacking. Although weighted stacking can be quite effective, as Schoenberger demonstrates, it does have a major disadvantage: only stacked traces are output. That limitation can be overcome by 2-D velocity filtering methods. For example, Ryu (1982) describes a filter applied to NMOcorrected CMP gathers in the space-time domain that separates multiples from primaries. The velocity function for the NMO lies between the velocity functions for primary events and multiple events. This, in effect, maps the events into different quadrants of the $f$ - $k$ domain, thereby making their separation possible by dip discrimination.

The next seven papers in this chapter describe multiple attenuation based on discrimination in a Radon transform domain. Currently this is the seismic exploration industry's most popular method of attenuating multiples. In a landmark paper, Hampson (1986) reports on multiple discrimination in the parabolic Radon transform domain. An NMO correction was applied to make the originally hyperbolic events in CMP gathers nearly parabolic in the $x-t$ domain, thereby mapping them into approximately discrete points after the parabolic transform. Theoretically, then, multiple discrimination is simple. Provided that a sufficient moveout difference existed originally between the two classes of events, they should map into separate regions in the Radon domain. In practice, unfortunately, things are not so simple. Usually the hoped for discrete points in the parabolic Radon domain are smeared out into overlapping zones of energy, making primary-multiple discrimination difficult. Since Hampson's 1986 paper, many authors have described refinements to his algorithm. Yilmaz (1989) improves on Hampson's method by replacing the NMO correction prior to the transform with a $t^{2}$-stretching of the CMP data. This converts the hyperbolic events to exact parabolas, improving the velocity resolution and hence the primary-multiple discrimination in the transform domain. Like Yilmaz, Foster and Mosher (1992) also set out to improve on Hampson's method. They describe suppression of multiples using the hyperbolic Radon transform rather than the parabolic transform. Foster and Mosher managed to find an "efficient hyperbolic surface" of integration for the transform stacking that was more accurate than the parabolic surfaces used by Hampson, but without additional computational cost.

In spite of the advances by Yilmaz and Foster and Mosher, Radon transforms were still sometimes a less than satisfactory way of discriminating between primary and multiple events. There were two basic problems:

- The finite spatial sampling and limited aperture in field data records limited resolution in the Radon domain (Thorson and Claerbout, 1985; Sacchi and Ulrych, 1995; Trad et al., 2003).
- In situations with complex geology the apexes of the hyperbolic events might not be at zero offset (as is assumed by a standard Radon transform).
The solution to the first problem was a so-called "high resolution" Radon transform, in which a priori statistical requirements were imposed that forced a sparse distribution of events in the Radon domain. Early forms of this type of transform required an expensive, iteratively re-weighted solution to a least-squares problem (see the references cited above). Herrmann et al. (2000) present a relatively inexpensive, non-iterative scheme to solve the problem. Their method is recursive; that is, the weights at each frequency depend on the Radon transform solution found at earlier frequencies. As an alternative, Moore and Kostov (2002) suggest a non-iterative, non-recursive scheme that derived the weights from semblance computations along the offset axis. Hargreaves et al. (2003) propose a solution to the second problem. Specifically, they address the attenuation of so-called diffracted multiples that arise from scatterers near the ocean bottom. Normal two-dimensional Radon transforms do a poor job of suppressing these multiples because their apexes do not occur at zero offset. Hargreaves et al. show that by adding a third Radon transform parameter, representing the apex location of hyperbolic events, superior separation of primaries and diffracted multiples is possible. Trad (2003) shows that a Stolt-type migration operator can be used to implement a very fast apex-shifted Radon transform that performs well in practice.

The design of the reject filter is a problem faced by all 2-D velocity filtering methods, regardless of the domain in which they operate. Ideally, the amplitudes of the transformed multiples and primaries would appear in well-localized and -separated regions so that the data processor could distinguish between these regions. In practice, the amplitudes are often not well-localized and the regions overlap, making selection of the reject filter difficult. Zhou and Greenhalgh (1996) solve this problem by using the 2-D transform of wave-equation-based multiple predictions (see Chapter 4) to design the optimal reject filters in the 2-D transform space. Zhou and Greenhalgh have published a series of papers showing that this method can be applied in any 2-D transform domain. Their paper included here discusses application to the parabolic Radon transform domain to attenuate water-layer multiples. Landa et al. (1999) extend Zhou and Greenhalgh's idea to attenuate both surface and interbed multiples.

Multiple reflections can be removed from seismic data by exploiting characteristics other than periodicity or moveout. For example, each multiple in a data set is kinematically and dynamically related to the primary events from the reflecting horizons involved in generating the multiple (see Chapter 4). The two papers by Doicin and Spitz (1991) and Manin and Spitz (1995) describe a multichannel pattern recognition technique that can target and eliminate a particular multiple based on its relationship with primary reflections. The 1991 and 1995 papers describe 2-D and 3-D applications of this idea, respectively. The method
requires knowledge of the generating mechanism of a targeted multiple.
Various forms of stacking and 2-D velocity filtering generally have the most difficulty separating primaries from multiples at near offsets, where the instantaneous apparent velocity differences between the two kinds of events typically vanish. Houston (1998) proposes enhancing the event discrimination at near offsets by applying a localized multichannel coherency filter to gathers NMO corrected with the moveout appropriate for multiples. If multiples are flattened, then they become laterally predictable, whereas overcorrected primary events are not. Houston's coherency filter appears to be more effective than $f$-k filtering in suppressing multiples without distorting primary reflections. Hu and White (1998) describe separating multiples from primaries using another type of coherency-based multichannel filtering called data-adaptive beamforming. This algorithm starts with a beamforming filter based on an initial model of coherent noise (i. e., a multiple) in a data set and then adaptively refines that model to optimize the ability of the beamforming filter to isolate that event. Hu and White show a prestack data example in which the performance of their method was superior to Radon transform-based multiple attenuation.

Finally, Zhang and Ulrych (2003) describe a method of separating primaries and multiples based on migration focusing. They first migrate the data prestack, using a velocity function appropriate for multiples, to focus the multiples, but not the primaries. Next, they apply a standard statistical measure, called the median of absolute deviations, along hyperbolic trajectories at the residual velocity of the primaries to identify samples that are outliers. Because only the multiples are focused, such outliers are almost certainly multiples. After replacing these samples with the median and demigrating, the multiples are significantly attenuated.

The papers in this chapter illustrate that attenuation methods based on differences between primaries and multiples are often an effective and appropriate choice within the toolbox of multiple attenuation techniques.

## Chapter 3 References

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## Chapter 4

## Multi-dimensional wavefield methods: Part I

In the volume introduction, we explained that every multiple event in a seismic data set, no matter how complicated, is a composite of two or more simpler events (hereafter called "subevents") that have their termination points at the free surface. This relationship suggests the possibility of manipulating a data set in a way that uses its subevents to predict its multiples. Indeed, Chapter 2 included several papers that discussed 1-D prediction of multiples from primaries. This chapter and the next include papers that carry this idea beyond 1-D to multi-dimensional wavefield manipulation methods that predict the multiple events in a data set. If such a prediction is sufficiently accurate, multiples can be eliminated simply by subtracting the predictions from the original data. This concept theoretically allows satisfactory multiple attenuation in situations where traditional methods, like those in Chapters 2 and 3, fail. In deep water, for example, simple periodicity-based methods fail because the multiples are not periodic. Furthermore, in recent years the search for petroleum targets has extended to geologic settings beneath complex heterogeneous overburdens such as salt, basalt, karsted sediments and volcanics. The lateral rapid heterogeneity and ill-defined boundaries in these settings are often too complex for any traditional methods of multiple attenuation to accommodate.

Three major methodologies for multiple prediction via multi-dimensional wavefield manipulation have evolved. The wavefield propagation method directly models multiples by propagating subevents through one or more cycles of reverberation (e.g., Wiggins (1988)). This approach requires a model of the medium, including velocities and reflection coefficients. Thus, it is useful mainly for water layer multiples since the required model information (water velocity and the water-bottom reflection coefficient) is relatively simple. For more complex reverberations, and especially for those in complex geologic settings, direct modeling is not usually considered a viable option because the required a-priori subsurface information is not known sufficiently well. Two alternative approaches, however, provide a capability of predicting multiples that avoid this problem. The feedback free-surface and interface model and the inverse-scattering series free-surface and point scatterer model are two distinct approaches for predicting both free-surface and internal multiples that reduce or eliminate the need for a-priori subsurface information. This is accomplished by using the data themselves to construct operators that predict the multiples contained within the data (e.g., Verschuur et al., (1992)). In particular, both methods can predict surface multiples without any need for a subsurface model. The two methods require knowledge of the acquisition wavelet in order to produce accurate multiple predictions. This chapter includes papers on the wavefield propagation and feedback methods; the inverse series procedures are found in Chapter 5. Because the basic concepts of these latter two methods are not widely known, we introduce them in a brief tutorial immediately following this chapter introduction.
In the first of the wavefield propagation papers, Bernth and Sonneland (1983) predict multiples by applying a water layer extrapolation operator in the frequency-wavenumber domain
to prestack data. The predicted multiples are adaptively subtracted from the original data to accommodate timing and amplitude errors in the prediction. Morley and Claerbout (1983) use a "Split-Backus" model to predict water layer peg-leg multiples. The modeling assumes near-vertical travel in the water layer, but can handle situations where the source and receiver depths are unequal and the water depth and water-bottom reflection coefficient vary along a line. The multiple modeling procedure suggested by Berryhill and Kim (1986) removes two important limitations present in the previous two papers. First, unlike the Bernth and Sonneland algorithm, the method can accommodate any sea-floor profile. Second, unlike the Morley and Claerbout procedure, the prediction is not limited to multiples that propagate nearly vertically in the water layer. The water-bottom reflection coefficient does not appear explicitly in Berryhill and Kim's multiple prediction procedure; instead, its effects are accounted for in an adaptive subtraction of the predicted multiples from the original data.

Wiggins (1988) derives a method of attenuating water-bottom multiples that allows for a locally varying water-bottom reflection coefficient. The wavefield propagation through the water layer is split into two pieces: a forward-in-time propagation from the surface to the water bottom and a backward-in-time propagation from the surface to the water bottom. The two wavefields at the water bottom then should be identical, trace-by-trace, except for the effects of the water-bottom and water-surface reflectivity. Minimizing the observed difference allows derivation of a filter representing the effects of the water-bottom reflectivity. That filter is then used to calculate the multiple-free wavefield. Lokshtanov (2000) describes a wave propagation method that is applied to CMP gathers in the $\tau-p$ domain. One advantage of this is that the method easily handles angular dependence of the water-bottom reflection coefficient. Finally, Hill et al. (2002) suggest predicting multiples using beam methods to extrapolate the wavefield. A predictive matched filtering to identify and remove multiples from the original data is applied to beam components. Using certain assumptions about the earth model, Hill et al. produce a form of their method that can be used with conventionally recorded 3-D marine data.

As explained in the Chapter 4 Tutorial, the feedback free-surface and interface model is a scheme for the forward modeling of seismic reflection data. When carried out in the seismic data processing, or inverse sense, it provides the opportunity to predict and attenuate multiples that are associated with the reflectors. Early versions of this concept can be found in the landmark works of Riley and Claerbout (1976), and Kennett (1979). Riley and Claerbout begin by using $Z$ - transforms to derive an algorithm that removes surface multiples for a 1-D earth model by, in essence, convolving the data with themselves. Their algorithm includes the inverse of the acquisition wavelet, and they describe how, in some cases, that wavelet can be found using a least-squares minimization of the difference between the seafloor primary convolved with itself and the first-order multiple water-bottom multiple. The paper then addresses 2-D multiple reflections. The authors derive an approximate finite difference-based solution that is analogous to their 1-D solution. The paper concludes with a lengthy discussion of the practical problems expected when the method is applied to less-than-ideal field data sets. Kennett describes construction of a surface multiple suppression operator in the frequency-wavenumber domain for plane-layered elastic and acoustic media.

The fundamental concept is the same as in the Riley and Claerbout paper, but Kennet's method is not restricted to wavefields that travel nearly vertically. Kennett also describes generalization of the method to land data. As is characteristic of the feedback free-surface methods, detailed knowledge of the acquisition wavelet is necessary.
In the early 1980's, Berkhout published a comprehensive treatment of the free-surface and interface model using a feedback formalism (Berkhout, 1982). In particular, Berkhout described an elegant, fully multi-dimensional $\omega-x$ formulation that could be described and implemented by simple matrix manipulations. Furthermore, the method placed no restrictions on the nature of the subsurface. This work launched a long-term effort, centered at Delft University, that addressed conceptual and practical issues and eventually brought this method - which is now known as "surface-related multiple elimination" (SRME) - to widespread industry usage. From among the numerous contributions from the Delft group, this chapter includes: Verschuur et al. (1992), Verschuur and Kabir (1992), Berkhout and Verschuur (1997), and Verschuur and Berkhout (1997). Verschuur et al. present the derivation of an $\omega$-x matrix equation for eliminating free-surface multiples. The equation requires no information about the subsurface, but it does require knowledge of the acquisition wavelet. Since that wavelet is typically not well known, the authors propose an adaptive procedure that estimates the wavelet by minimizing the energy in the data after multiples are removed. The matrix equation includes the inverse of a large matrix, which is computationally expensive and may have stability issues. To overcome these problems, Verschuur et al. do a series expansion of the matrix inverse and keep only a few lower-order terms. The authors also mention briefly a recursive scheme for attenuating internal multiples as well (see below). Verschuur and Kabir make a comparison between surface-related multiple elimination and Radon transform multiple elimination. They conclude that the two methods complement each other. This paper also includes a simplified version of the SRME theory, which makes it a good starting point for readers new to the concept. The Berkhout and Verschuur and Verschuur and Berkhout papers are companions: the first is concerned with theory, and the second with practical issues and examples. In these two papers, the authors explain and illustrate an iterative version of the SRME method.
A reciprocity formulation of this free-surface and interface model for multiple removal also had its historical roots and activity centered in Delft University, from the school of Professor A. deHoop. Pioneers like Fokkema and Van den Berg (1990) developed these concepts within a wave-theoretical integral equation framework, thereby providing mathematical clarity and physical insight. This approach furthered the understanding of the relationship between the feedback and inverse-scattering methods (see Chapter 5) for free-surface multiple attenuation, and the role, for example, that the obliquity factor (Born and Wolf, 1964) plays in those theories. The wave theoretical angle-dependent obliquity factor is important, particularly for long offsets and shallow targets, as illustrated by three figures from an EAGE Convention paper by Dragoset (1993). The figures, which do not appear in the original published abstract, are included at the end of this chapter introduction.
In addition to those from the Delft group, this chapter includes a few other notable papers on the feedback free-surface and interface model. In the early 1980's Pann filed for a US

Patent, Pann (1989), for a multi-dimensional method of predicting and removing surface multiples that is based on Huygens' principle. Numerically, the steps in Pann's method for predicting multiples of a certain order for a specific trace are the same as those in the Delft scheme. Pann, however, did not reveal any method for easily selecting which combinations of traces need to be convolved. The matrix formulation of the Delft method accomplishes that task automatically. Dragoset and Jericevic (1998) derive the equations for SRME in an intuitive fashion by making an analogy between surface multiple prediction and the diffraction aperture problem of classical optics. The Kirchhoff integral solves both problems, and the authors show how the integration can be accomplished by matrix manipulations. They also present and discuss a list of survey design suggestions intended to provide data that are most suitable for the SRME process. Finally, Al-Bannagi and Verschuur (2003) propose a method of applying SRME to post-stack data. For a given post-stack trace, the data are demigrated to produce sufficient pre-stack traces for predicting the multiples in the (zero offset) post-stack trace. This approach has several advantages when applying SRME to land data:

- The multiple prediction is performed with traces that have good signal-to-noise ratio compared to that of pre-stack land traces.
- Problems with irregular and sparse spatial sampling of the surface wavefield are avoided.
- Detailed structural variations in the subsurface are accommodated.
- The method is computationally efficient.

As several of the papers in this chapter discuss, the feedback free-surface and interface model method can be applied to the prediction and attenuation of internal multiples. Starting with the free surface and proceeding to the next shallowest reflector, the water bottom, one sequentially predicts and removes first all multiples having downward reflections at the free surface and then all multiples having their shallowest downward reflection at the water bottom. Continuing in this manner, all free-surface and internal multiples are attenuated one interface at a time. Carrying out this strategy requires accurate depth migration to each interface and good estimation of the reflectivity properties at that imaged reflector. However, extension of the composite event concept to internal multiples allows certain important characteristics of those multiples to be predicted in a simpler fashion. In particular, all internal multiples from a given interface can have their travel times predicted precisely without any knowledge of the subsurface except for the location in time of that reflector. Amplitudes, however, are predicted only approximately with an accuracy that depends on the type of internal multiple. For further discussion of various issues involving internal multiple prediction see the tutorial following this introduction as well as Coates and Weglein (1996) and Weglein and Matson (1998) in Chapter 5. Here, we include one paper, Jakubowicz (1998), that describes the prediction of internal multiples based on the composite event concept and the free-surface and interface model.
The Delft University feedback formulation for multiple prediction has met with much success, and, in fact, has become an industry-wide standard method of removing multiples. Other related formulations have also led to viable wave equation-based methods of predicting and attenuating multiples. This chapter concludes with one such effort: Liu et al. (2000). These
authors derive multiple attenuation formulas using the invariant embedd
This chapter concludes with one such effort: Liu et al. (2000). These authors derive multiple attenuation formulas using the invariant embedding approach. The results resemble those that appear in the Delft papers. However, the method is implemented in the $\tau-p$ domain. For media with only gentle dips, the resulting algorithm is quite efficient because the matrices involved in the multiple prediction are sparse.

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Figure 4: Figure A: Synthetic marine shot record for testing SRME algorithms. The model has three flat reflecting horizons in a constant velocity medium. The zero-offset arrival times for the three primary events are about 0.5, 0.7, and 1.2 s . The reflection coefficients were chosen such that the third primary event (at the arrow) is exactly cancelled by one of the surface-related multiples. The strong event at 1.0 s and all of the events below 1.2 s are surface-related multiples. Figure B. Surface-related multiples eliminated - no obliquity factor. This result was obtained by directly inverting the matrix at each frequency that represents the SRME operator. That operator is derived from the Kirchhoff integral (Dragoset and Jericevic, 1998). Here, the obliquity factor part of the Kirchhoff integral was ignored by setting it equal to one for all wavefront angles of incidence at the surface. Figure C. Surface-related multiples eliminated - proper obliquity factor applied. Compare this result to that in Figure B. Using the proper obliquity factor improves multiple elimination at the large offsets. This is to be expected, because the raypaths are most oblique to the surface at large offsets.

## Chapter 5

## Multi-dimensional wavefield methods: Part II

The free-surface and point scatterer model for the generation of seismic reflection data provides a free-surface model for free-surface multiple generation and a point scatterer model for generating primaries and internal multiples. When this model operates in an inverse sense, the free-surface model removes ghosts and free-surface multiples while the point-scatterer model allows for processing primaries to produce structure maps, earth property estimates, and the removal of internal multiples. The history of this approach derives from a form of perturbation theory called scattering theory and describes how altering (or perturbing) a medium will result in an altered or perturbed wave-field. The tremendous flexibility in scattering theory allows using either a surface, interface or point scatterer model to characterize the difference between the original (reference) and the perturbed medium (actual earth) depending on ones ability to provide or define the difference between reference and earth in an inverse or processing sense. For example, since the air-water boundary is fairly well defined, a free-surface description for that perturbation is chosen and that model is then used to generate and remove free-surface multiples. Since typical subsurface detail is much less well defined, a point scatterer description of the perturbation is chosen for generating and processing primaries and internal multiples.

The origin of the inverse scattering series is found in atomic and nuclear scattering (e.g., Moses, 1956) and was extended to acoustics, (e.g. Prosser (1969), Razavy (1975), and then to seismic exploration by Weglein, Boyse and Anderson (1981). Convergence problems and other practical issues precluded the series, in that pristine form, from providing any practical value. To extract some practical usefulness from this most general and flexible formalism, Weglein, Carvalho, Araujo, and Stolt sought to separate the series into task-specific subseries resulting in distinct algorithms for attenuating free-surface and internal multiples, and to investigate their convergence and practical requirements. Carvalho et al. (1991, 1992) developed the free-surface inverse-scattering subseries and then applied it successfully to synthetic data (1991) and to field data (1992). Araujo et al. (1994) first identify the subseries that attenuates internal multiples, and exemplify with tests that include freesurface and internal multiples. Weglein et al. (1997) present the first comprehensive theory for attenuating all multiples from a multi-dimensional heterogeneous earth with absolutely no information about the subsurface. The excellent convergence properties of these subseries and their ability to accommodate field data were in marked contrast to properties of the overall series.

Coates and Weglein (1996) examine and test the efficacy of prediction of the amplitude and phase of internal multiples for acoustic and elastic media. The phase of all internal multiples is correctly predicted, including that of converted-wave multiples, and the predicted amplitude well attenuates internal multiples of an entire P-wave history. Weglein and Matson (1998) use an analytic example to understand the precise nature of the amplitude predicted
in the internal multiple algorithm and provide a sub-event interpretation for the time of the predicted internal multiple phase. Matson (1996) provides a map between the forward construction of seismic events in the scattering series and the primaries and multiples in seismic data.

A series describes primaries and multiples in terms of reference propagation and repeated point scatterer interactions where the scattering from any point depends on the difference between actual and reference medium properties at that point. The inverse processes on primaries and multiples require only reference propagation and reflection data. The individual terms in the inverse series that remove internal multiples attenuate all internal multiples of a given order from all interfaces at once - without interpretive intervention or event picking of any kind - and automatically accommodate multiples due to specular, corrugated or diffractive origin. The leading term in the removal series predicts the time of internal multiples precisely; higher-order terms increase the accuracy of the amplitudes of the predicted multiples, thereby allowing better attenuation.

The collaborative project between the feedback group at Delft University and the inverse scattering practitioners at Atlantic Richfield Company helped the latter recognize that providing a range of depths where the troublesome internal multiple at the target might have its downward reflection, allows a truncation of an inner integration in the inverse scattering internal multiple attenuation algorithm, with concomitant significant computational savings. Although the interface method is computational efficient when the reflector causing the downward reflection can be isolated, the ability to automatically attenuate internal multiples under complex geologic conditions without requiring any subsurface information, isolation of reflectors, or interpretive intervention, remains the unique strength of the inverse scattering method of attenuating internal multiples.

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## Chapter 6

## Multiple prediction for field data sets

Both the multiple prediction method developed at Delft University and by others (Chapter 4) and the method based on scattering theory (Chapter 5) are complete and realistic algorithms from a multidimensional physics point of view. Theoretically, such completeness and realism allow accurate multiple prediction without requiring detailed a priori subsurface information. However, while these methods can accommodate complex wave propagation effects without knowledge of the subsurface, they do place stringent requirements on the definition and completeness of the seismic measurements at the surface. Typically, those requirements are not fully met by present day routine data acquisition practices. This chapter contains papers that describe efforts to overcome the problems that occur when multiples are predicted using less than ideal surface wavefield measurements.

There are two types of stringent requirements. (The Chapter 4 tutorial and Dragoset and Jericevic (1998), also in Chapter 4, discuss the reasons for these requirements.) First, the acquisition wavelet must be either measured or estimated accurately for each shot in the recorded data set. The acquisition wavelet consists of the source signature, the source and receiver array radiation patterns including surface ghost effects, and the recording system filters. Note that the acquisition wavelet is angle dependent and that it does not include any earth filtering effects. Second, the wavefield measured at the surface must be fully sampled and non-aliased. Fully sampled means the following: Suppose a particular recorded trace is defined by its shot and receiver locations at the surface. To predict surface multiples for that trace requires that its shot also be recorded by a 2-D spread of receivers that spans all possible surface locations at which the various multiples in the trace may have their downward reflections. Furthermore, the trace's receiver must record data from a 2-D spread of shots that spans the same surface locations. The size of the aforementioned 2-D spreads that is, the surface recording aperture - depends on the subsurface structure. For example, if the structure is predominantly 2-D and seismic lines are shot parallel to the dip direction, then 1-D spreads and 2-D multiple prediction suffice. If, however, the subsurface contains 3-D structures - such as ocean-bottom diffractors, reflecting horizons with crossline dip, and salt structures - then 2-D spreads and 3-D multiple prediction are required. Generally, the size of the required crossline aperture is a function of crossline dip.

The requirement for an accurate acquisition wavelet was an important obstacle to the early acceptance and application of wave equation-based multiple prediction technology. An early response to that challenge sought to exploit the multiple attenuation algorithm's sensitivity to having the wavelet by using the algorithm itself to find its own required wavelet. The basic assumption was that seismic data without multiples had fewer events and hence less energy. Therefore, the desired wavelet could be found by searching for the wavelet that produced an energy minimum when multiples predicted using that wavelet were subtracted from the original data. Various incarnations of this concept were introduced. Verschuur et al. (1992)
(see Chapter 4) propose a preprocessing deconvolution to remove the angle-dependent components of the acquisition wavelet from the data and follow that with a frequency-dependent energy-minimizing search to determine the residual wavelet amplitude and phase. Carvalho and Weglein (1994) describe a global and robust searching of the minimum-energy objective function surface designed to avoid local minima. Ikelle et al. (1997) truncate the scattering series formulation of multiple prediction after two terms, which results in a linear relationship between the acquisition wavelet and the free-surface reflections. Ignoring truncation errors, the resulting energy-minimization problem for the wavelet has an analytic solution (that can be refined through an iterative scheme to account for truncation effects). Dragoset and Jericevic (1998) (see Chapter 4) formulate the surface multiple attenuation algorithm using a closed-form expression of the multiple prediction series. Unfortunately, that formulation results in the expression for the acquisition wavelet being inside of a rather large matrix that must be inverted, making an iterative wavelet search quite expensive. They show, however, that by applying eigenvalue decomposition to the matrix the acquisition wavelet factor can be isolated in a diagonal matrix, which, of course, is easily and cheaply inverted.

Basically, all of these methods for estimating the acquisition wavelet work at cross-purposes to the underlying physics of the multiple attenuation methods they are meant to serve. For example, destructively interfering events cause no problem for the physics of the algorithms but can cause problems with minimum-energy wavelet estimation approaches. There are other approaches to finding the acquisition wavelet, such as near-field source measurements (Ziolkowski et al., 1982), that are independent of the multiple attenuation process. Ziolkowski, et al. (1999) propose a wave-theoretical multiple attenuation algorithm that uses near-field source measurements to avoid three problems common to most methods of Chapters 4 and 5 . They are: 1) the assumption that the source is a point, 2) the presence of the incident source field in the recorded data, and 3) the need to estimate the wavelet using a minimum-energy criterion. (Note: being a wavefield method, the Ziolkowski, et al., 1999 paper could have been included in Chapter 4. However, its emphasis on dealing with problems in field data sets makes it at home in this chapter as well.) A variant of Green's theorem known as the extinction theorem provides another possibility for satisfying the need to know the acquisition wavelet (Weglein et al., 2000).

Prior to the year 2002, the attenuation methods of Chapters 4 and 5 were typically implemented as 2-D algorithms, and applied to data that were, at best, quasi 3-D. In the presence of 3-D subsurface structures, the resulting timing errors in predicted multiples can be quite large; see, for example, Ross et al. (1999). Furthermore, such timing errors are complicated; they depend on crossline dip, offset, and the order of the multiples. The obvious solution to this problem was to apply 3-D prediction algorithms to 3 -D data, but costs generally prohibited this. Therefore, a more pragmatic solution was sought for and developed: sophisticated adaptive subtraction. The general idea behind this approach is that because of imperfections in field data sets and use of prediction algorithms that ignore 3-D complications, effective multiple attenuation can seldom be accomplished in just one step. Instead,
a two-step method is necessary: (1) the multiples are imperfectly predicted followed by (2) adaptive subtraction that compensates for the imperfections, thereby producing reasonably good attenuation in spite of them. Note that adaptive subtraction compensates for prediction errors that arise due to imperfect knowledge of the acquisition wavelet as well as the errors due to use of 2-D rather than 3-D multiple prediction.

Adaptive subtraction can be applied to different data domains (e.g., common shot, common offset, etc.) using many different algorithms, all of which have parameter settings that can affect the results. This flexibility is a mixed blessing: it allows for good results in a wide variety of situations, but the data processing practitioner can face an overwhelming smorgasbord of choices. Using synthetic data, Abma et al. (2002) compare the performance of least-squares 1-D matching filters, pattern-matching (Spitz, 1999), and shaped 2-D filters. They conclude that, of those three choices, 1-D matching filters (computed from and applied to a 2-D window of data) are the safest to use for adaptive subtraction. The other two methods, while theoretically more accommodating of errors in the predicted multiples, tend to attenuate primary reflections along with the multiples. van Borselen et al. (2003) present a target-oriented adaptive subtraction approach. Their method uses the same 1-D matching algorithm as studied by Abma et al., but it is applied to subtract only multiples having some particular characteristic, such as a predominant dip or frequency range. Presumably, such a constraint minimizes the chance that the adaptive subtraction process will effect primary reflections. Guo (2003) describes a more advanced pattern-matching algorithm than that studied by Abma et al. It iteratively calculates the prediction error filter for the primaries and uses a projection signal filter to reduce the effects of random noise. Guo's results on the simple synthetic data sets analyzed by Abma et al. look superior to results of other methods applied to those data sets. In a somewhat different approach, Ross et al. (1999) suggest applying time-variant deterministic time corrections to the predicted multiples prior to the adaptive subtraction. The time corrections are derived from a 3-D model of the subsurface.

The need to accommodate errors in multiples produced by applying 2-D prediction to data from a 3-D subsurface was one main impetus behind the development of adaptive subtraction as part of the multiple attenuation process. Although the method of 2-D prediction followed by adaptive subtraction has had many successes, its limitations are also evident. Consider, for example, the offshore Norway data set results displayed by Hadidi et al. (2002). A simple CMP stack (Figure 4) shows a semi-coherent noisy region that is produced by diffracted surface multiples. A 2.5-D application of surface multiple attenuation (see the paper for an explanation of $2.5-\mathrm{D}$ ) produced good results when the source-to-receiver crossline separation was small and noticeably poorer results when the source-to-receiver crossline separation was large (Figures 6 and 13, respectively). In neither case was the incoherent noise caused by the diffracted multiples completely attenuated. This result is not a surprise since diffracted multiples are an inherently 3 -D phenomenon. The inability to attenuate them fully is due to limitations of current data acquisition practices.
Although possible remedies to the limitations of current data acquisition practices were envisioned at about the same time as when adaptive subtraction became widely used, prac-
tical applications of those ideas have appeared in the literature only recently. Two types of remedies are possible. Although full 3-D marine acquisition with towed streamers may never be practical, there are novel acquisition schemes that offer benefits. Alternatively, data acquired with standard survey designs may be extrapolated and interpolated in various ways and at various processing stages to simulate full 3-D data acquisition. Keggin et al. (2003) present a simple, but expensive, acquisition remedy for the problems created by 3-D diffracted multiples. Using an 8-cable streamer ship and a separate shooting ship, a single target swath was acquired nine times with different source-receiver configurations. Stacking the resulting multi-azimuth data sets produced a significant reduction in the noise due to diffracted multiples.

If a 3-D surface multiple prediction algorithm is applied to standard 3-D marine streamer data the sparse crossline sampling of the surface wavefield causes the predicted multiples to be a poor representation of the actual multiples. The next two papers, van Dedem and Verschuur (2001) and Hokstad and Sollie (2003) attack this problem using sparse inversion (hyperbolic and parabolic, respectively) applied to the crossline multiple contribution gathers. (These gathers consist of the collection of traces that are summed to produce a trace containing predicted multiples.) Specifically, the inversion produces a parametric representation of the information in a sparsely populated crossline multiple contribution gather from which accurate predicted multiples are calculated as if the gather were fully populated. Nekut (1998) offers a different solution to the sparse crossline sampling problem of streamer acquisition. He proposes using least-squares migration-demigration as an interpolation and extrapolation process to create a fully populated 3-D data set from standard 3-D field measurements. Although computationally expensive, small-scale synthetic data set tests suggest that the method has promise. The final paper in this chapter, Kleemeyer et al. (2003), is the first exploration industry publication to describe actual application of 3-D surface multiple attenuation to an entire 3-D marine streamer survey. Multiple prediction was accomplished by Shell's MAGIC3D algorithm (Biersteker, 2001), which includes a massive data regularization, extrapolation, and interpolation effort. Interestingly, in Kleemeyer et al. the predicted multiples are subtracted from the data after both were prestack depth migrated. The results show superior attenuation of diffracted multiples compared to earlier processing efforts.

As this reprint volume goes to press, we think that the seismic industry is just at the beginning of a major effort to develop practical, cost-effective ways of applying 3-D surface multiple prediction to 3-D marine streamer surveys. Although the theory and underlying physical basis of the method are well understood, many pragmatic compromises will likely be necessary and remain to be discovered. While deterministic prediction of multiples brings greater effectiveness, such predictions will never be perfect; hence, there will always be a role for statistical and adaptive procedures. These issues also motivate the drive for more effective data collection, such as single sensor data, and extrapolation and interpolation methods. The heightened demand on definition and completeness will be increasingly satisfied in the coming years, leading to a new level of effectiveness for the attenuation of free surface and internal multiples, and the subsequent imaging and inversion methods for primaries.

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## Chapter 7

## Multiple attenuation for land, ocean bottom, and vertically-deployed marine receiver data sets

Interestingly, most of the technical literature on multiple attenuation addresses the problem of multiple reflections in marine streamer data. Perhaps, arguing that the relatively high reflection coefficients of the water surface and the water bottom make multiples in marine data more of a problem than those in land data, we should laud geophysicists for tackling the tougher problem. On the other hand, one could counter that their very subtlety makes multiples in land data the tougher problem of the two. Regardless of the outcome of such an argument, it is certainly true that the non-uniformity and noise contamination of typical land data sets makes removing multiples a difficult, expensive problem indeed. Furthermore, because the source of multiples in a land data set is rarely obvious, data processors are often left with a nagging uncertainty: Were the events affected by that process really multiples, or were they perhaps primary reflections?
Kelamis et al. (1990) describe a practical methodology for applying the Radon transform (Yilmaz (1989), Chapter 3) to attenuate multiples in high-density land data sets. By using a pre-processing step to reduce the multiplicity and regularize the geometry, they managed to obtain convincing results at a modest cost. Ten years later, Kelamis and Verschuur (2000) investigated the application of surface-related multiple elimination (SRME) (Verschuur et al. (1992), Chapter 4) to land data sets. SRME was originally conceived as a method of eliminating surface multiples in marine data, where the surface has a nearly uniform reflection coefficient. To make this method applicable for land data, Kelamis and Verschuur balanced the data amplitudes in a pre-processing step so as to have smooth multiple prediction operators. They obtained good multiple suppression in land data sets for which other methods (moveout discrimination and predictive deconvolution) were unsatisfactory.

Technologies for recording exploration-quality seismic data on the sea floor have become viable only since about 1990. The main difficulty with sea-floor recording (other than hardware issues) is the impact of the receiver ghost reflection and subsequent water-column reverberations on the data bandwidth. Many years ago, J. E. White (1965) proposed a solution to that problem for pressure measurements. Currently, the seismic industry has many schemes based on White's idea to record both pressure and the vertical component of particle velocity and combine those measurements to cancel reverberations in the water column. Barr (1997) presents an overview of this method and Barr et al. (1997) compare several different variations of the method. Amundsen et al. (1998) describe a generalization of the concept: up/down splitting based on the elastodynamic representation theorem. Unlike earlier methods, their algorithm is valid for a dipping sea floor with medium parameters that vary laterally. Osen et al. (1999) extended White's concept to remove water-layer multiples from multicomponent sea-floor data, including the horizontal components of particle velocity. Finally, Amundsen (2001) describes an algorithm, based on up/down wavefield
separation, that removes the effect of the free surface entirely as well as accomplishing signature deconvolution. Up/down wavefield separation requires two recordings: either pressure and the vertical component of particle velocity or pressure and its vertical derivative. The method also requires measurement of the direct arrival from the marine source.

Although marine data sets recorded by towed streamers and ocean-bottom sensors differ in many ways, they do have one characteristic in common: the receivers are deployed in a horizontal or near-horizontal plane. Sonneland et al. (1986) describe using data recorded with two horizontal marine streamers, one above the other in the vertical plane, to accomplish receiver deghosting. Their algorithm is based on up/down wavefield separation, and thus, as in Amundsen (2001) allows both dereverberation and designature. An alternative vertical acquisition scheme is to deploy entire marine cables vertically, a method developed during the 1990's. As with the borehole VSP method, vertical marine cables allow separation of the recorded pressure wavefield into its up- and down-going components. That capability by itself does not, however, solve the multiple problem. Wang et al. (2000) describe attenuating multiples in vertical cable data using a three-step process: a common-shot tau-p filter to remove receiver ghost multiples, common-receiver deconvolution to remove source ghosts, and a Radon filter to remove other multiples.

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## Chapter 8

## Tutorials, reviews and case histories

This chapter represents a collection of papers that provide: (1) a perspective of the broad landscape of techniques and approaches to attenuating multiples, e.g., papers (26,66), (2) a detailed description of a particular method, plus an attempt to bridge to another approach with similar objectives but with a different language and history, seeking to better understand and differentiate between substantive (rather than cosmetic) differences and strengths/weaknesses, e.g., papers ( $60,43,67,7$ and 68 ) and (3) case studies that exemplify different techniques with field data comparisons and where relative effectiveness is described and plausibility arguments proposed to explain these differences, e.g., papers $(44,75)$.

Basically, there has been a rejuvenated interest in multiple attenuation over the past few years- due to the industry trend to deep water, with a collection of technical and economic challenges driving the need for reduced risk and increased reliability. A direct response to this challenge has come from new thinking, testing, development and application on multidimensional wave theoretic multiple removal techniques that do not require knowledge of the subsurface. This moves the boundary between deterministic and signal processing/statistical methods into the region formally occupied by the latter. There are, and always will be, aspects of the recorded reflection data that are outside any chosen deterministic modeland progress and effectiveness in attenuating multiples on field data is determined by how well we can address that critical component, as well. Sometimes those non-deterministic approaches are explicit, with sophisticated statistical models, e.g., to model the difference between signal and random noise. Other times the procedures are implicit and subtler with a call for adaptive and data example specific parameter estimation: a recognition that the chosen deterministic model needs some user intervention and assistance, to accommodate model inadequacy or prerequisite satisfaction.

As a general guideline and guiding principle to choosing multiple removal algorithms, a tool box approach seems appropriate -where the approach is chosen by a set of factors including: (1) geologic complexity of multiple generating reflectors: horizontal, deep, specular, corrugated and diffractive, (2)cost/benefit, (3) seismic processing objective, and (4) availability of data, algorithm and satisfaction of practical prerequisites.

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## Chapter 9

## Multiples as signal

There are two basic views of seismic reflection data. The inclusive view treats all reflection data as signal, since all events contain information about the subsurface. In contrast, the exclusive view treats only primary reflections as signal and considers multiples an undesirable noise. The latter view - that espoused by the first eight chapters of this volume - by far dominates, for a good reason. Extracting accurate depth images from just primary reflection information is, in general, a complex, difficult task. Conventional wisdom is that imaging with multiples must be significantly more challenging. As the papers in this chapter illustrate, however, challenging conventional wisdom can lead to surprising and promising ideas. Specifically, extracting useful information from multiple reflections may not be as difficult as traditionally thought.
Reiter et al. (1991) demonstrate the use of receiver-side, first-order surface multiples to enhance migrated primary reflection images in data recorded on the ocean bottom. This was accomplished by using moveout discrimination to isolate the multiples of interest and then migrating them using a ray-based Kirchhoff depth migration. The resulting composite image had improved signal-to-noise ratio and extended lateral subsurface coverage. In contrast to Reiter et al., Berkhout and Verschuur (1994) propose a general scheme for migrating all surface multiples in a data set rather than just those with a specific ray path. The method requires isolation of the surface multiple component of the seismic response and the formation of areal source wavefields using the total seismic response. The authors note that the method can be extended to internal multiples, although no details are given. Sheng (2001) suggests yet another method of using the information in surface multiples. First-order surface multiples are cross-correlated with primaries to remove the first leg of their raypath. The remaining second leg is migrated using the appropriate primary migration operator. Youn and Zhou (2001) describe a migration scheme that migrates a source function in the forward direction and the recorded traces in a shot record in the backward direction, using - for both cases - a full two-way scalar wave equation. The two propagated wavefields are correlated and summed over all time indices to produce an image frame. The claim is that this scheme uses all types of events to produce a depth migrated image directly from raw field records. Because of the massive data storage and computational requirements, the authors applied the algorithm to only an extremely decimated version of the synthetic Marmousi model. Finally, Berkhout and Verschuur (2003) introduce the notion of a "focal transform," a variation on the feedback model method of multiple prediction (see Chapter 4) that involves correlation rather than convolution. The focal transform reduces the order of each surface multiple in a data set by one. This allows primary information to be extracted from multiples. One application is the extracting of short-offset primary reflections (that are not directly recorded) from longer-offset multiples (that are directly recorded).
The papers in this chapter clearly support the premise that the multiple wavefield in a seismic data set contains useful information about the earth's subsurface. Thus, arguably a better
approach to the multiple problem is using that information rather than discarding it. Based on the papers included here, data correlation seems to be a key to making use of multiples. However, use of multiply reflected energy is still a relatively unexplored terrain. Other means of exploiting, rather than attenuating, multiple energy may remain to be discovered. We expect further efforts in this area in the years ahead.

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# A leading order imaging series for prestack data acquired over a laterally invariant acoustic medium. Part I: Derivation and preliminary analysis 

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#### Abstract

As a multidimensional direct inversion procedure, the inverse scattering series has the ability to image reflectors at their correct spatial location using only the reflection data and an approximate velocity model as input. Therefore, it is a good candidate procedure for deriving an algorithm that can accurately depth image seismic data in complex geological areas where the velocity model is difficult to estimate.

Previously, a subseries of the inverse series was isolated that images reflectors in space without requiring the actual propagation velocity of the medium. This leading order imaging series was found to converge for large contrasts between the actual and reference medium for a 1-D medium and a normal incident plane wave experiment. In this paper (Part I), in preparation for an analysis for data missing low temporal frequency (Part II), the algorithm is formulated for an experiment in which a point source explodes in a three-dimensional constant density acoustic medium where the velocity varies only in depth. The formulation, termed "prestack" in reference to the degree of freedom that exists due to the source-receiver offset, is parameterized for constant angles of incidence.

The leading order imaging series is an approximation to the full depth imaging potential of the inverse series in that it is leading order in the data. The first term in the series images reflectors at the depths dictated by the constant reference velocity and the data's travel times. The remaining terms use the data's amplitudes and travel times as well as the reference velocity to shift the reflectors closer to their correct location in depth.

Analytic and numerical examples are used to demonstrate that, for small contrasts between the actual and reference medium, the leading order imaging series significantly improves the predicted depths of the reflectors at precritical angles, effectively flattening the angle gathers. For higher contrasts, or when greater accuracy is desired, then higher order imaging terms are required that go beyond the leading order terms identified and analyzed in this paper.


## 1 Introduction (motivation and background)

Depth imaging of seismic reflection data plays a critical role in the exploration and production of oil and gas. The primary goal of depth imaging is to produce a spatially accurate map of reflectors below the earth's surface. This structural map is important to the oil and gas industry because it plays a key role in determining where to drill for hydrocarbon reserves, which has an enormous economic, environmental and political impact.

Current depth imaging algorithms can be formulated from a linear inverse scattering model, in which the reference velocity is assumed to be close enough to the actual velocity in order to place reflectors at their correct spatial locations. In practice, especially in complex geological environments, the most accurate methods for deriving the reference velocity model may be inadequate for linear imaging algorithms inasmuch as they fail to focus the reflection energy at the correct location. The inverse scattering series has the ability to image primary reflection events at their correct location using only the reflection data and an approximate reference velocity model (Weglein et al., 2000). The first term in the inverse series is a linear inversion of the data. It uses a velocity model that is incorrect below the measurement surface to image reflectors at locations expected when imaging with a typical depth imaging algorithm (essentially with the equation depth $=$ velocity $\times$ travel time). Therefore, in general, the first term in the series will mislocate the reflectors unless the velocity model is correct. The higher order terms in the inverse series, that are non-linear in the data, contain parts that move the reflectors to their true spatial locations. These non-linear imaging terms only exist when the velocity model is incorrect. In fact, the inverse series only exists when the reference and actual media are different.

As a multidimensional direct inversion procedure, the inverse scattering series has more to do than image reflectors at their correct locations in space. The inverse series removes multiply reflected events, images reflectors, and inverts amplitudes for medium parameters directly using only the measured data and a reference medium's parameters (Moses, 1956; Razavy, 1975; Stolt and Jacobs, 1980; Weglein et al., 1981). Early numerical tests of the inverse series' ability to directly invert seismic data (Jacobs, 1980; Carvalho, 1992) suggested that it converged only when the reference medium properties were very close to the actual medium properties. Multiples were a significant impediment to estimating the earth's properties (and deriving an adequate reference medium) from seismic reflection data. As a result, research was undertaken into using the inverse series to derive algorithms that removed free-surface and internal multiples from seismic data, but that stopped short of imaging and parameter estimation. The strategy employed was to isolate subseries of the inverse series that are responsible for removing multiples.

Weglein et al. (1997) have derived multidimensional free surface and internal multiple attenuation algorithms by isolating separate subseries of the inverse series that perform these two tasks. These subseries turned out to have more favorable convergence properties than the entire inverse series. The algorithms converge for an acoustic reference medium of water and they share the advantage of not requiring information about the earth below the
measurement surface. An important prerequisite of all inverse series algorithms is that the source wavelet is known. These multiple attenuation algorithms are now routinely used in seismic processing to remove multiples prior to velocity estimation, imaging and AVO analysis. A key concept within the subseries approach to inversion of seismic data, is to apply the task-specific algorithms in the following order: 1.) free surface multiple removal; 2.) internal multiple removal; 3.) depth imaging and 4.) target identification. In taking this staged approach, each step is less ambitious than direct inversion for earth properties and is therefore likely to be less demanding of the input data, reference medium proximity, and of computational requirements. In addition, the subseries are allowed to benefit from all of the tasks that have been achieved earlier in the sequence, thus further simplifying the problem at each step. For a comprehensive review of the inverse scattering series and its application to seismic exploration, see Weglein et al. (2003).

The strategy employed in developing inverse scattering subseries algorithms to solve problems in seismic data processing has been to first consider the simplest situation in which the particular problem exists. Most often this is a 1-D normal incidence experiment in a constantdensity acoustic medium. The simplest reference medium is chosen that agrees with the actual medium above the measurement surface and confines the perturbation to be below the receivers. Then the inverse series is analytically computed and a subseries is sought that is responsible for achieving the specific processing task (one of the four listed above). The subseries is isolated through a combination of intuition and experience garnered through studying the forward series terms that construct the seismic wavefield.

If the subseries algorithm demonstrates an intrinsic ability to achieve its objective, then it is reformulated and generalized so that it may eventually be tested on multidimensional field data. Shaw et al. (2003) considered the simple situation of a normal-incidence experiment over a 1-D constant density acoustic medium for which the velocity was an unknown function of depth. An imaging series algorithm was derived that imaged reflectors in depth using a constant reference velocity and it was shown analytically that this series converged for large finite contrasts between the actual and reference velocities. For relatively small contrasts, the leading order imaging series is a good approximation to the entire imaging series in that the predicted depths are a significant improvement over linear depth imaging with the reference velocity. It was also demonstrated that this series converges more quickly for smaller contrasts and for lower maximum frequencies. Therefore, a proximate reference velocity and a source spectrum with a lower maximum frequency both aid the rate of convergence.

Having established for the simplest examples that the leading order imaging series has good convergence properties, the next stage in developing a practical algorithm is to evaluate its performance under increasingly realistic conditions. Since seismic data are always frequency bandlimited, one of the highest priority tests involves an analysis of the algorithm under conditions of missing low frequencies. Such an analysis is provided by Shaw and Weglein (2004) in a second paper (Part II). In preparation for that analysis, the leading order imaging series algorithm is rederived here to accommodate prestack input data. The offset aperture in prestack data provides a lower vertical wavenumber $k_{z}$ and more closely resembles the actual seismic experiment. This paper includes a derivation of the prestack leading order imaging
series and presents some preliminary analytic and numerical examples. These examples are used to discuss how the imaging series performs the task of depth imaging given a constant reference velocity that is never updated.
We consider a 3-D constant density acoustic medium with point sources and receivers located at $\vec{x}_{s}=\left(x_{s}, y_{s}, z_{s}\right)$ and $\vec{x}_{g}=\left(x_{g}, y_{g}, z_{g}\right)$, respectively. Wave propagation in this medium can be characterized by the wave equation

$$
\begin{equation*}
\left(\nabla^{2}+\frac{\omega^{2}}{c^{2}\left(\vec{x}_{g}\right)}\right) P\left(\vec{x}_{g} \mid \vec{x}_{s} ; \omega\right)=-A(\omega) \delta\left(\vec{x}_{g}-\vec{x}_{s}\right) \tag{1}
\end{equation*}
$$

where $P$ is the pressure field, $A$ is the source wavelet, $c$ is the propagation velocity and $\omega$ is the angular frequency. The temporal Fourier transform is defined by

$$
\begin{equation*}
P\left(\vec{x}_{g} \mid \vec{x}_{s} ; \omega\right)=\int_{-\infty}^{\infty} P\left(\vec{x}_{g} \mid \vec{x}_{s} ; t\right) e^{i \omega t} d t \tag{2}
\end{equation*}
$$

To simplify the current analysis of the prestack imaging series, we will be assuming that the medium varies only in the $z$ direction. For the generalization to a 2-D earth, see Liu et al. (2004). The velocity, $c$, can be expressed in terms of a constant reference velocity, $c_{0}$, and a perturbation, $\alpha$, such that

$$
\begin{equation*}
\frac{1}{c^{2}(z)}=\frac{1}{c_{0}^{2}}(1-\alpha(z)) \tag{3}
\end{equation*}
$$

For this acoustic problem, the goal of inversion is to solve for $\alpha$ which can be written as an infinite series

$$
\begin{equation*}
\alpha=\alpha_{1}+\alpha_{2}+\alpha_{3}+\cdots \tag{4}
\end{equation*}
$$

where $\alpha_{1}$, the first term in the series for $\alpha$, is linearly related to the scattered field, $P_{s}=$ $P-P_{0} . P_{0}$ is the pressure wavefield due to the same source, $A(\omega)$, in the reference medium, i.e., a wholespace with velocity, $c_{0}$. The second term, $\alpha_{2}$, is quadratic in $P_{s}$, the third term, $\alpha_{3}$, is cubic and so on. After using the inverse series (4) to solve for $\alpha$, we can use (3) to solve for the unknown velocity, $c(\vec{x})$. The objective of the research described here is in fact not to solve for the medium parameters (in this case just $c$ ), but to solve directly for the location at which the perturbation $\alpha$ changes. This is the problem of imaging in a medium whose velocity is not known before or after the imaging procedure.

## 2 A leading order imaging series for a 3-D experiment over a laterally invariant acoustic medium

### 2.1 The first term, $\alpha_{1}$, and its degree of freedom

If the source wavelet is deconvolved so that $\tilde{D}=P_{s} / A$, then $\tilde{D}$ is related to $\alpha_{1}$ by

$$
\begin{equation*}
\tilde{D}\left(\vec{x}_{g} \mid \vec{x}_{s} ; \omega\right)=\int_{-\infty}^{\infty} G_{0}\left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(\vec{x}^{\prime}\right) G_{0}\left(\vec{x}^{\prime} \mid \vec{x}_{s} ; \omega\right) d \vec{x}^{\prime} \tag{5}
\end{equation*}
$$



Figure 1: Plan view showing the relationship between the horizontal cartesian and cylindrical coordinates. $r$ is the source-receiver offset in the horizontal plane and $\phi$ is the azimuth. For a 1-D subsurface, the data are invariant in azimuth.
where $k_{0}=\omega / c_{0}$ and $G_{0}$ is the causal Green's function satisfying the wave equation in the reference medium. The solution for $\alpha_{1}$ in cylindrical coordinates is (see Appendix A)

$$
\begin{equation*}
\tilde{\alpha}_{1}\left(-2 q_{g}\right)=2 \pi \frac{-4 q_{g}^{2}}{k_{0}^{2}} e^{i q_{g}\left(z_{g}+z_{s}\right)} \int_{0}^{\infty} \tilde{D}(r ; \omega) J_{0}\left(k_{r} r\right) r d r \tag{6}
\end{equation*}
$$

where the vertical and horizontal wavenumbers, $q_{g}$ and $k_{r}$, respectively, are related by

$$
\begin{equation*}
q_{g}=\frac{\omega}{c_{0}} \sqrt{1-\frac{k_{r}^{2} c_{0}^{2}}{\omega^{2}}} \tag{7}
\end{equation*}
$$

$J_{0}\left(k_{r} r\right)$ is a zero order Bessel function of the first kind that arises due to the azimuthal symmetry and is

$$
\begin{equation*}
J_{0}\left(k_{r} r\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{i k_{r} r \cos \phi^{\prime}} d \phi^{\prime} \tag{8}
\end{equation*}
$$

Figure 1 illustrates the relationship between the horizontal cartesian coordinates $\left(x_{g}, y_{g}\right)$ and cylindrical coordinates $(r, \phi)$.

The fact that the data are a function of both time and source-receiver offset whereas $\alpha$ is only a function of depth is evident in (6) in that $\tilde{\alpha}_{1}$ is over-determined. Whereas $\tilde{\alpha}_{1}$ is only a function of $q_{g}$, the right-hand side of (6) can be written as a function of two independent variables, e.g., $\left(q_{g}, \omega\right)$ or $\left(k_{r}, \omega\right)$. Large angles of incidence can construct $\tilde{\alpha}_{1}$ at low $q_{g}$ values since $q_{g}=k_{0} \cos \theta_{0}$. Inverse Fourier transforming both sides of (6) gives

$$
\begin{align*}
\alpha_{1}(z) & =\frac{2}{2 \pi} \int_{-\infty}^{\infty} \tilde{\alpha}_{1}\left(-2 q_{g}\right) e^{-2 i q_{g} z} d q_{g} \\
& =-8 \int_{-\infty}^{\infty} \frac{q_{g}^{2}}{k_{0}^{2}} e^{-i q_{g}\left(2 z-\left(z_{g}+z_{s}\right)\right)} \int_{0}^{\infty} \tilde{D}(r ; \omega) J_{0}\left(k_{r} r\right) r d r d q_{g} \tag{9}
\end{align*}
$$

where $q_{g}^{2} / k_{0}^{2}=\cos ^{2} \theta_{0}$. Considering fixed angles of incidence, $\theta_{0}$, leads to a number of different estimates of $\alpha_{1}$, denoted by $\alpha_{1}\left(z, \theta_{0}\right)$. Fixing $\theta_{0}$ is the same as fixing horizontal and vertical slownesses, $p_{0}$ and $\zeta_{0}$, respectively, where

$$
p_{0}=\frac{\sin \theta_{0}}{c_{0}} \text { and } \zeta_{0}=\frac{\cos \theta_{0}}{c_{0}} .
$$

However, $q_{g}$ is still allowed to vary through the variation in $\omega$ since $q_{g}=\omega \zeta_{0}$. We proceed by changing variables from $q_{g}$ to $\omega$ :

$$
\begin{equation*}
\alpha_{1}\left(z, \theta_{0}\right)=-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{-\infty}^{\infty} e^{-i \omega \zeta_{0}\left(2 z-\left(z_{g}+z_{s}\right)\right)} \int_{0}^{\infty} \tilde{D}(r ; \omega) J_{0}\left(\omega p_{0} r\right) r d r d \omega \tag{10}
\end{equation*}
$$

Defining $\tau_{0}=\zeta_{0}\left(2 z-\left(z_{g}+z_{s}\right)\right)$ and performing the inverse temporal Fourier transform of the data $\tilde{D}(r ; \omega),(10)$ becomes

$$
\begin{equation*}
\alpha_{1}\left(z, \theta_{0}\right)=-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{0}^{2 \pi} \int_{0}^{\infty} D\left(r ; \tau_{0}-p_{0} r \cos \phi\right) r d r d \phi \tag{11}
\end{equation*}
$$

Changing back to cartesian coordinates yields (see Appendix A)

$$
\begin{equation*}
\alpha_{1}\left(z, \theta_{0}\right)=-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D\left(x_{g}, y_{g} ; \tau_{0}-x p_{0}\right) d x_{g} d y_{g} \tag{12}
\end{equation*}
$$

Equation (12) is recognizable as a scaled slant stack of the recorded data (Treitel et al., 1982). In cartesian coordinates, it requires sums in both the $x$ and $y$ directions, whereas in cylindrical coordinates, as a result of the symmetry of a laterally invariant medium, these integrals are replaceable by integrals over $\phi$ and $r$ (11) or $\omega$ and $r$ (10). An alternative approach to handling the degree of freedom in (9) is to hold $\omega$ fixed and integrate over angle or vertical slowness $\zeta_{0}=\cos \theta / c_{0}=q_{g} / \omega$. This parameterization will result in different estimates of $\alpha_{1}$ for constant $\omega$ values and is the subject of ongoing research.

### 2.2 Task separation in the second and third terms

The integral equation for the second term in the inverse series for this acoustic problem is

$$
\begin{align*}
\int_{-\infty}^{\infty} G_{0}\left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right) & k_{0}^{2} \alpha_{2}\left(\vec{x}^{\prime}\right) G_{0}\left(\vec{x}^{\prime} \mid \vec{x}_{s} ; \omega\right) d \vec{x}^{\prime} \\
= & -\int_{-\infty}^{\infty} d \vec{x}^{\prime} G_{0}\left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(\vec{x}^{\prime}\right) \\
& \times \int_{-\infty}^{\infty} d \vec{x}^{\prime \prime} G_{0}\left(\vec{x}^{\prime} \mid \vec{x}^{\prime \prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(\vec{x}^{\prime \prime}\right) G_{0}\left(\vec{x}^{\prime \prime} \mid \vec{x}_{s} ; \omega\right) \tag{13}
\end{align*}
$$

The solution to (13) is detailed in Appendix B where $\tilde{\alpha}_{2}$ as a function of vertical wavenumber is shown to be

$$
\begin{equation*}
\tilde{\alpha}_{2}\left(-2 q_{g}\right)=-\int_{-\infty}^{\infty} d z^{\prime} e^{2 i q_{g} z^{\prime}} \frac{k_{0}^{2}}{2 q_{g}^{2}}\left(\alpha_{1}^{2}\left(z^{\prime}\right)+\int_{0}^{z^{\prime}} d z^{\prime \prime} \alpha_{1}\left(z^{\prime \prime}\right) \frac{d \alpha_{1}\left(z^{\prime}\right)}{d z^{\prime}}\right) . \tag{14}
\end{equation*}
$$

As in the case of $\tilde{\alpha}_{1}$, there is a degree of freedom in (14) that results in a choice of which variable to hold constant, and which to integrate over in the construction of $\alpha_{2}(z)$. For example, if we choose to keep incident angle $\theta_{0}$ constant, then performing the inverse Fourier transform of (14) gives

$$
\begin{equation*}
\alpha_{2}\left(z, \theta_{0}\right)=-\frac{1}{2 \cos ^{2} \theta_{0}}\left(\alpha_{1}^{2}\left(z, \theta_{0}\right)+\int_{0}^{z} d z^{\prime} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) \frac{\partial \alpha_{1}\left(z, \theta_{0}\right)}{\partial z}\right) \tag{15}
\end{equation*}
$$

since

$$
\frac{k_{0}^{2}}{q_{g}^{2}}=\frac{k_{0}^{2}}{k_{0}^{2} \cos ^{2} \theta}=\frac{1}{\cos ^{2} \theta} .
$$

The second term in the inverse series (15) has been separated into the sum of two pieces. As explained by Weglein et al. (2002), these two terms have distinctly different roles in the inversion procedure. The first piece

$$
\begin{equation*}
\alpha_{21}\left(z, \theta_{0}\right)=-\frac{1}{2 \cos ^{2} \theta_{0}} \alpha_{1}^{2}\left(z, \theta_{0}\right) \tag{16}
\end{equation*}
$$

is responsible for correcting the amplitude of $\alpha_{1}\left(z, \theta_{0}\right)$ (see, e.g., Zhang and Weglein, 2004) and the second piece

$$
\begin{equation*}
\alpha_{22}\left(z, \theta_{0}\right)=-\frac{1}{2 \cos ^{2} \theta_{0}} \int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime} \frac{\partial \alpha_{1}\left(z, \theta_{0}\right)}{\partial z} \tag{17}
\end{equation*}
$$

acts to shift the mislocated interfaces in $\alpha_{1}\left(z, \theta_{0}\right)$ closer to their true depths. This shift is accomplished by a Taylor series for the difference of two Heaviside functions expanded about the depth of each mislocated interface. The first term in this Taylor series for the shift is $\alpha_{22}$. For details on this expansion, the reader is referred to Weglein et al. (2002).
Proceeding to the third term in the series, the integral equation to be solved is

$$
\begin{align*}
\int_{-\infty}^{\infty} d \vec{x}^{\prime} G_{0} & \left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right) k_{0}^{2} \alpha_{3}\left(z^{\prime}\right) G_{0}\left(\vec{x}^{\prime} \mid \vec{x}_{s} ; \omega\right) \\
= & -\int_{-\infty}^{\infty} d \vec{x}^{\prime} G_{0}\left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d \vec{x}^{\prime \prime} G_{0}\left(\vec{x}^{\prime} \mid \vec{x}^{\prime \prime} ; \omega\right) k_{0}^{2} \alpha_{2}\left(z^{\prime \prime}\right) G_{0}\left(\vec{x}^{\prime \prime} \mid \vec{x}_{s} ; \omega\right) \\
& -\int_{-\infty}^{\infty} d \vec{x}^{\prime} G_{0}\left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right) k_{0}^{2} \alpha_{2}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d \vec{x}^{\prime \prime} G_{0}\left(\vec{x}^{\prime} \mid \vec{x}^{\prime \prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(z^{\prime \prime}\right) G_{0}\left(\vec{x}^{\prime \prime} \mid \vec{x}_{s} ; \omega\right) \\
& -\int_{-\infty}^{\infty} d \vec{x}^{\prime} G_{0}\left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d \vec{x}^{\prime \prime} G_{0}\left(\vec{x}^{\prime} \mid \vec{x}^{\prime \prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(z^{\prime \prime}\right) \\
& \times \int_{-\infty}^{\infty} d \vec{x}^{\prime \prime \prime} G_{0}\left(\vec{x}^{\prime \prime} \mid \vec{x}^{\prime \prime \prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(z^{\prime \prime \prime}\right) G_{0}\left(\vec{x}^{\prime \prime \prime} \mid \vec{x}_{s} ; \omega\right) . \tag{18}
\end{align*}
$$

The solution for $\alpha_{3}(z)$ can be broken into a number of pieces two of which are given in Appendix C. This separation is discussed in Shaw et al. (2003) for the normal incidence case and it is extendable to prestack data when the angle $\theta_{0}$ is held constant. The amplitude correction and leading order imaging contributions from the third term are given by:

$$
\begin{equation*}
\alpha_{3}\left(z, \theta_{0}\right)=\frac{1}{\cos ^{4} \theta_{0}}\left[\frac{3}{16} \alpha_{1}^{3}\left(z, \theta_{0}\right)+\frac{1}{8}\left(\int_{0}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}\right)^{2} \frac{\partial^{2} \alpha_{1}\left(z, \theta_{0}\right)}{\partial z^{2}}+\cdots\right] . \tag{19}
\end{equation*}
$$

### 2.3 A prestack leading order imaging series

The imaging series is a subseries of the inverse series that is responsible for positioning reflectors at their correct spatial location (Weglein et al., 2000, 2002). For the problem considered here, in which the earth is characterized by a single parameter, the imaging series is written

$$
\begin{equation*}
\alpha^{\mathrm{IM}}=\alpha_{1}^{\mathrm{IM}}+\alpha_{2}^{\mathrm{IM}}+\alpha_{3}^{\mathrm{IM}}+\cdots \tag{20}
\end{equation*}
$$

where $\alpha_{i}^{\mathrm{IM}}$ is the term in the imaging series that is $i^{\text {th }}$ order in the scattered field and is found in the $i^{\text {th }}$ term of the inverse series. The leading order imaging series, $\alpha^{\text {LOIM }}$, is the contribution to the imaging series that is leading order in the scattered field. The terms in this imaging series have been found to exhibit a specific pattern (corresponding to particular scattering diagrams) recognized by Shaw et al. (2003) which allowed the prediction of a general form. Following that earlier work, and using the constant- $\theta_{0}$ formulation, the prestack form of the algorithm is

$$
\begin{align*}
\alpha^{\mathrm{LOIM}}\left(z, \theta_{0}\right)= & \alpha_{1}\left(z, \theta_{0}\right)-\frac{1}{2} \frac{1}{\cos ^{2} \theta_{0}}\left(\int_{0}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}\right) \frac{\partial \alpha_{1}\left(z, \theta_{0}\right)}{\partial z} \\
& +\frac{1}{8} \frac{1}{\cos ^{4} \theta_{0}}\left(\int_{0}^{z} \alpha_{1}\left(z, \theta_{0}\right) d z^{\prime}\right)^{2} \frac{\partial^{2} \alpha_{1}\left(z, \theta_{0}\right)}{\partial z^{2}}-\cdots \\
= & \sum_{n=0}^{\infty} \frac{(-1 / 2)^{n}}{n!\cos ^{2 n} \theta_{0}}\left(\int_{0}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}\right)^{n} \frac{\partial^{n} \alpha_{1}\left(z, \theta_{0}\right)}{\partial z^{n}} \tag{21}
\end{align*}
$$

where

$$
\alpha_{1}\left(z, \theta_{0}\right)=-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{0}^{2 \pi} \int_{0}^{\infty} D\left(r ; \tau_{0}-p_{0} r \cos \phi\right) r d r d \phi
$$

Performing a Fourier transform of (21) gives

$$
\begin{align*}
\alpha^{\mathrm{LOIM}}\left(k_{z}, \theta_{0}\right) & =\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{\left(-i k_{z} /\left(2 \cos ^{2} \theta_{0}\right)\right)^{n}}{n!}\left(\int_{0}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}\right)^{n} \alpha_{1}\left(z, \theta_{0}\right) e^{-i k_{z} z} d z \\
& =\int_{-\infty}^{\infty} \exp \left[-i k_{z} /\left(2 \cos ^{2} \theta_{0}\right) \int_{0}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}\right] \alpha_{1}\left(z, \theta_{0}\right) e^{-i k_{z} z} d z \tag{22}
\end{align*}
$$

where the power series

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{\left[-i k_{z} /\left(2 \cos ^{2} \theta_{0}\right) \int_{0}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}\right]^{n}}{n!}=\exp \left[-i k_{z} /\left(2 \cos ^{2} \theta_{0}\right) \int_{0}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}\right] \tag{23}
\end{equation*}
$$

was substituted to arrive at (22). Inverse Fourier transforming both sides of (22) yields a closed form for the leading order imaging series that operates $\theta_{0}$-by- $\theta_{0}$ (or $p_{0}$-by- $p_{0}$ ):

$$
\begin{equation*}
\alpha^{\mathrm{LOIM}}\left(z, \theta_{0}\right)=\alpha_{1}\left(z-1 /\left(2 \cos ^{2} \theta_{0}\right) \int_{0}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}, \theta_{0}\right) . \tag{24}
\end{equation*}
$$

The prestack leading order imaging series for a point source in a laterally invariant acoustic medium can be implemented by slant-stacking (or $\tau-p$ transforming) the data, weighting each $p$ trace by the factor $-8 \zeta_{0} \cos ^{2} \theta_{0}$, and then operating on each trace with the formula provided in (24). When $\theta_{0}=0,(21)$ and (24) reduce to the normal incidence algorithms given by Shaw et al. (2003). The normal incidence leading order imaging series was shown to converge for arbitrarily large finite contrasts between the actual and reference medium. From (23), it can be concluded that the rate of convergence of (21) will be greater for smaller values of $k_{z}$, smaller values of $\int_{0}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}$, and smaller values of $\theta_{0}$. Analysis of the 1-D normal incidence algorithm showed that, for relatively small contrasts (the actual velocity within about $10 \%$ of the reference velocity), the leading order contributions to the imaging series can accurately locate reflectors. Higher contrasts or greater accuracy require higher order imaging terms.

## 3 Analytic and numerical examples

### 3.1 Analytic example with two interfaces; leading and higher order imaging terms

Consider a model that consists of two horizontal interfaces at depths $z_{a}$ and $z_{b}$ and a discontinuous velocity profile $c_{0}-c_{1}-c_{2}$ (Fig. 2). The wavefield in the upper halfspace, $\Psi_{0}$, consists of an incident field, $\Psi_{i}$, and a reflected field, $\Psi_{r}$. The measured reflected wavefield can be derived by decomposing the incident field into a sum of plane waves (the Sommerfeld integral) and then matching boundary conditions at each interface. The result is

$$
\begin{equation*}
\Psi_{r}\left(r, z_{g} ; \omega\right)=i \omega \int_{0}^{\infty} \frac{\left(R_{01}+T_{01} R_{12} T_{10} e^{2 i \omega \zeta_{1}\left(z_{b}-z_{a}\right)}+\cdots\right)}{\zeta_{0}} e^{i \omega \zeta_{0}\left(2 z_{a}-z_{s}-z_{g}\right)} J_{0}\left(\omega p_{0} r\right) p_{0} d p_{0} \tag{25}
\end{equation*}
$$

where the reflection and transmission coefficients are functions of angle and are given by

$$
\begin{equation*}
R_{01}=\frac{\zeta_{0}-\zeta_{1}}{\zeta_{0}+\zeta_{1}}, T_{01}=\frac{-2 \zeta_{1}}{\zeta_{0}+\zeta_{1}}, R_{12}=\frac{\zeta_{1}-\zeta_{2}}{\zeta_{1}+\zeta_{2}} \text { and } T_{10}=\frac{2 \zeta_{0}}{\zeta_{0}+\zeta_{1}} . \tag{26}
\end{equation*}
$$



Figure 2: A multi-layer 1-D constant density acoustic model. In the absence of a free surface, all reflected waves at the receiver are upgoing.

We further define $R_{12}^{\prime}=T_{01} R_{12} T_{10}$. The vertical slownesses are functions of the incident angles in each layer since

$$
\begin{equation*}
\zeta_{i}=\frac{\cos \theta_{i}}{c_{i}}, i=0,1,2, \ldots \tag{27}
\end{equation*}
$$

The " $+\cdots$ " in (25) are the internal multiple reflections in the data. The internal multiple removal subseries, that begins in the third term of the inverse series, is assumed to have been applied before the imaging subseries. This results in a new effective data and a new effective $\alpha_{1}$ that contain only primary reflection events. This step is part of the strategy of inverse series task separation described by Weglein et al. (2003). For the two reflector example considered here, the internal multiples are of no consequence since the imaging series only uses information recorded earlier than the primary event being imaged, which excludes the multiples. Reverting to the symbol $\tilde{D}$ for data that contain only primary reflections, and changing the integration variable from $p_{0}$ to $k_{r}$, (25) becomes

$$
\begin{equation*}
\tilde{D}(r ; \omega)=-\int_{0}^{\infty} \frac{\left(R_{01}+R_{12}^{\prime} e^{2 i \omega \zeta_{1}\left(z_{b}-z_{a}\right)}\right)}{i \omega \zeta_{0}} e^{i \omega \zeta_{0}\left(2 z_{a}-z_{s}-z_{g}\right)} J_{0}\left(k_{r} r\right) k_{r} d k_{r} \tag{28}
\end{equation*}
$$

Substituting the data (28) into the linear inverse equation (10), then for this two-reflector
example, the first term in the series for $\alpha(z)$ can be written as a function of angle

$$
\begin{align*}
\alpha_{1}\left(z, \theta_{0}\right) & =-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{-\infty}^{\infty} e^{-i \omega \zeta_{0}\left(2 z-\left(z_{g}+z_{s}\right)\right)} \int_{0}^{\infty} \tilde{D}(r ; \omega) J_{0}\left(k_{r} r\right) r d r d \omega \\
& =8 \cos ^{2} \theta_{0} \int_{-\infty}^{\infty} \frac{R_{01}+R_{12}^{\prime} e^{2 i \omega \zeta_{1}\left(z_{b}-z_{a}\right)}}{i \omega \zeta_{0}} e^{-2 i \omega \zeta_{0}\left(z-z_{a}\right)} d \omega \\
& =4 \cos ^{2} \theta_{0}\left[R_{01} H\left(z-z_{a}\right)+R_{12}^{\prime} H\left(z-z_{b^{\prime}}\right)\right] \tag{29}
\end{align*}
$$

where the shallower reflector is correctly located at $z_{a}$ (since the velocity down to $z_{a}$ was correct) but the deeper reflector is mislocated at depth

$$
\begin{equation*}
z_{b^{\prime}}=z_{a}+\left(z_{b}-z_{a}\right) \frac{\zeta_{1}}{\zeta_{0}} . \tag{30}
\end{equation*}
$$

Therefore, the correction in the depth of the second reflector from $z_{b^{\prime}}$ to the actual depth $z_{b}$ is

$$
\begin{align*}
z_{b}-z_{b^{\prime}} & =z_{b}-z_{a}-\left(z_{b}-z_{a}\right) \frac{\zeta_{1}}{\zeta_{0}} \\
& =\left(z_{b}-z_{a}\right)\left(1-\frac{\zeta_{1}}{\zeta_{0}}\right) \tag{31}
\end{align*}
$$

Substituting (30) into (31) to eliminate $z_{b}$ on the right-hand side gives

$$
\begin{equation*}
z_{b}-z_{b^{\prime}}=\left(z_{b^{\prime}}-z_{a}\right)\left(\frac{\zeta_{0}}{\zeta_{1}}-1\right) \tag{32}
\end{equation*}
$$

The ratio of the slownesses can be written as an infinite series in the reflection coefficient (or amplitude) of the shallower reflector under the condition that $\left|R_{01}\right|<1$. This condition precludes post-critical reflections. Therefore,

$$
\frac{\zeta_{0}}{\zeta_{1}}=\frac{\left(1-R_{01}\right)}{\left(1+R_{01}\right)}=1-2 R_{01}+2 R_{01}^{2}-2 R_{01}^{3}+\cdots,\left|R_{01}\right|<1
$$

Hence, the shift from the depth predicted by the first term in the series $\left(z_{b^{\prime}}\right)$ to the actual depth of the second reflector $\left(z_{b}\right)$ can be expressed, for precritical angles, as

$$
\begin{equation*}
z_{b}-z_{b^{\prime}}=-2\left(z_{b^{\prime}}-z_{a}\right)\left(R_{01}-R_{01}^{2}+R_{01}^{3}-\cdots\right) . \tag{33}
\end{equation*}
$$

The approximation to this shift that is leading order in the data's amplitudes is

$$
\begin{equation*}
z_{b}-z_{b^{\prime}} \approx-2\left(z_{b^{\prime}}-z_{a}\right) R_{01} . \tag{34}
\end{equation*}
$$

This is equal to the shift calculated automatically by the leading order imaging series. To see this, we substitute the first term in the imaging series for this example (29) into the
closed form for $\alpha^{\text {LOIM }}(24)$ and evaluate the algorithm at $z_{b^{\prime}}$

$$
\begin{align*}
\alpha^{\mathrm{LOIM}}\left(z_{b^{\prime}}, \theta_{0}\right) & =\alpha_{1}\left(z_{b^{\prime}}-1 /\left(2 \cos ^{2} \theta_{0}\right) \int_{0}^{z_{b^{\prime}}} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}, \theta_{0}\right) \\
& =\alpha_{1}(z_{b^{\prime}}-2 \int_{0}^{z_{b^{\prime}}} R_{01}\left(\theta_{0}\right) H\left(z^{\prime}-z_{a}\right) d z^{\prime}-2 \underbrace{\int_{0}^{z_{b^{\prime}}} R_{12}^{\prime}\left(\theta_{0}\right) H\left(z^{\prime}-z_{b^{\prime}}\right) d z^{\prime}}_{=0}) \\
& =\alpha_{1}\left(z_{b^{\prime}}-2\left(z_{b^{\prime}}-z_{a}\right) R_{01}\left(\theta_{0}\right)\right) . \tag{35}
\end{align*}
$$

We see from (35) that the leading order imaging series $\alpha^{\text {LOIM }}$ shifts the interface at $z_{b^{\prime}}$ in $\alpha_{1}$ to a new depth $z_{b^{\prime}}+2\left(z_{b^{\prime}}-z_{a}\right) R_{01}$ which is closer to the actual depth $z_{b}$. As might be expected, this shift is a function of angle.

The extent to which the leading order imaging series, $\alpha^{\text {LOIM }}$, is a good approximation to the entire imaging series, $\alpha^{\mathrm{IM}}$, depends on the magnitude of the perturbation above the reflector being imaged. Higher order imaging series that go beyond the leading order approximation include successively more amplitude terms in the series for the shift in (33). For models containing more than two interfaces, the leading order order imaging series produces an approximation to the shift at each mislocated interface that is an infinite series in reflection and transmission coefficients in the overburden. It is postulated that higher terms in the imaging series will act to unravel these transmission coefficients.

Consider two specific examples where the reference velocity $c_{0}=1500 \mathrm{~m} / \mathrm{sec}$ and the two reflectors are located at $z_{a}=1000 \mathrm{~m}$ and $z_{b}=1075 \mathrm{~m}$. In the first case $c_{1}=1650 \mathrm{~m} / \mathrm{sec}$ and in the second case $c_{1}=1350 \mathrm{~m} / \mathrm{sec}$. Figure 3 illustrates the depths predicted by the first term in the series and three approximations to the imaging series for two different velocity models. The variation of $z_{b^{\prime}}$ with angle is referred to as residual moveout. At higher angles, the depth of the second reflector predicted by the first term in the series is less accurate. This is because the constituent plane waves travelling at higher angles of incidence spend a proportionally longer time in the layer with the wrong velocity. Therefore, the non-linear terms in the imaging series have to shift the interface further at higher angles. The fact that the magnitude of the reflection coefficient at the first interface, $\left|R_{01}\right|$, increases with angle aids the imaging terms in shifting greater distances with angle. On the other hand, this increase in amplitude will tend to make the leading order approximation in (34) less justifiable. Figure 3 shows, for two examples, that including higher order imaging terms improves the accuracy of the predicted depth, especially at higher angles where they are needed more. Figure 4 shows two more examples where the contrasts are twice as large as in Fig. 3. These examples show how higher order imaging terms become more important for higher contrasts between the actual and reference velocity.


Figure 3: Low contrast analytic example. Depths predicted by the first term in the series and three different imaging series as a function of angle for two specific analytic examples: $z_{a}=1000 m, z_{b}=1075 m, c_{0}=1500$ $\mathrm{m} / \mathrm{sec}$ and $c_{1}=1650 \mathrm{~m} / \mathrm{s}$ (i), $c_{1}=1350 \mathrm{~m} / \mathrm{sec}$ (ii).


Figure 4: High contrast analytic example. Depths predicted by the first term in the series and three different imaging series as a function of angle for two specific analytic examples: $z_{a}=1000 m, z_{b}=1075 m, c_{0}=1500$ $\mathrm{m} / \mathrm{sec}$ and $c_{1}=1800 \mathrm{~m} / \mathrm{sec}(i), c_{1}=1200 \mathrm{~m} / \mathrm{sec}$ (ii).

### 3.2 Numerical examples

We test the leading order imaging series on data synthesized using a reflectivity algorithm (see, for example, Kennett (1983)). The source wavelet is a band-limited delta function with a frequency spectrum $A(f)$ where $f_{\min }<f<f_{\max }$. We begin with the simplest imaging problem of two reflectors in a constant density acoustic medium with no free surface.

As with the analytic examples, two specific cases are considered, one representing the case where the reference velocity is slower than the actual velocity, and one where the reference velocity is faster. In the former example, the velocities are $c_{0}=1500 \mathrm{~m} / \mathrm{s}, c_{1}=1650 \mathrm{~m} / \mathrm{s}$ and $c_{2}=1500 \mathrm{~m} / \mathrm{s}$, and so the critical angle for a downgoing plane wave at the first interface is $65^{\circ}$. In the latter example, the velocities are $c_{0}=1500 \mathrm{~m} / \mathrm{s}, c_{1}=1350 \mathrm{~m} / \mathrm{s}$ and $c_{2}=$ $1500 \mathrm{~m} / \mathrm{s}$, and there is no critical reflection. The depths of the two interfaces in both examples are $z_{a}=1000 \mathrm{~m}$ and $z_{b}=1075 \mathrm{~m}$.

The data are synthesized in the $\tau-p$ domain and so can be considered to have been generated from an experiment with infinite spatial aperture. Figure 5 shows the reflectivity data for the two models. In both cases, the minimum and maximum source frequencies are $f_{\text {min }}=0.25$ Hz and $f_{\max }=62.5 \mathrm{~Hz}$, respectively. We choose to display the result as "spike-like" data, rather than "box-like" $\alpha_{1}$, by taking the derivative of the result with respect to $z$. This is done primarily because it is easier to detect the shifting of reflectors when displayed in "spike-like" form. Figure 6 shows the results of imaging the data in Fig. 5 using the constant reference velocity, $c_{0}$. The mislocated reflector exhibits residual moveout when imaged with the first term in the series (left). The leading order imaging series (right) improves the depth at all angles and acts to "flatten" the imaged reflector. As expected from the leading order approximation, a small amount of residual moveout remains.

Figure 7 compares the result of summing eight terms in the leading order imaging series (21) with the closed form result (24). After summing eight terms, the series has converged and the deeper reflector has been relocated to the depth predicted analytically. Summing more terms in the leading order imaging series does not further correct the depth because the terms are too small. Differences in appearance between the series summation and the closed form result can be attributed to the artifacts associated with the numerical computation of derivatives.

Figure 8 shows the velocity profile and synthetic data for a 6 -layer model. The imaging results using a constant reference velocity are compared in Fig. 9. The leading order imaging series improves the location of all the reflectors mislocated by the first term. The remaining errors in the predicted depths are left to be corrected by higher order imaging terms. A small cumulative error in depth noticeable in Fig. 9 is attributed primarily to the fact the integral of the data in the overburden necessarily includes transmission coefficients that introduce small errors in the series for the shifts.


Figure 5: Velocity model and synthetic reflectivity data in the $\tau$-p domain for two specific two-interface examples. The time derivatives of the data are displayed and the polarity is consistent with the wave equation in (1). The red lines overlying the seismic data are the analytically computed $\tau$ values for each reflector.

## 4 Discussion

The analytic and numerical results of the prestack leading order imaging series have highlighted a number of interesting characteristics of the algorithm. Given a choice of how to handle the degree of freedom afforded by the source-receiver offset in the seismic experiment, we chose in this paper to keep the angle of incidence in the reference medium ( $\theta_{0}$ or $p_{0}$ ) constant and let $\omega$ vary. This allowed for a straightforward generalization of the normal incidence case to non-normal incidence. By parameterizing the problem for a constant $\theta_{0}$, one can consider a new effective perturbation that is scaled by $1 / \cos ^{2} \theta_{0}$ :

$$
\begin{equation*}
\frac{\omega^{2}}{c^{2}(z)}=\frac{\omega^{2}}{c_{0}^{2}}(1-\alpha(z))=\frac{k_{z}^{2}}{\cos ^{2} \theta_{0}}(1-\alpha(z)) . \tag{36}
\end{equation*}
$$

For the acoustic examples considered here, the first term in the imaging series more accu-


Figure 6: Results of imaging the two datasets in Fig. 5. At top is the example where the velocity increased, and at bottom is the example where the velocity decreased. On the left is the first term in the series: the result of an imaging algorithm that is linear in the data. On the right is the result of the leading order imaging series. The derivatives with respect to depth of $\alpha^{\text {LOIM }}$ are displayed. The yellow lines are the actual depths of the two reflectors. The red and green lines are the predicted depths computed analytically using (30) and (34), respectively.
rately locates reflectors at small angles $\theta_{0}$. This is because the decomposed plane waves at larger angles spend proportionally longer times in the layers with the wrong velocity. At the same time, the magnitude of the amplitudes at larger angles is greater, which assist the non-linear terms of the leading order imaging series in shifting the reflectors the required distance to their actual locations. This larger "effective contrast" at higher angles also tends to emphasize the fact that the imaging algorithm currently being tested is leading order in the data's amplitudes. This leading order approximation is better at small angles and is it's the reason why the leading order imaging series leaves a small amount of residual moveout. Higher order terms may be required by larger angles of incidence. On the other hand, the


Figure 7: The cumulative sum of up to eight terms in the leading order imaging series compared to the closed form result. The input data are the same as in Fig. 5 (top) where the reference velocity is slower than the actual velocity in the layer. The red and green lines are the predicted depths computed analytically for the first term and the closed form, respectively.
rate of convergence of the series form of the algorithm will benefit from the fact that the maximum $k_{z}$ is smaller at large angles (since $k_{z}=k_{0} \cos \theta_{0}$ ).


Figure 8: Velocity profile and synthetic reflectivity data in the $\tau-p$ domain for a 6 layer model. The time derivatives of the data are displayed. The red lines overlying the seismic data are the analytically computed $\tau$ values for each reflector.

The reflectivity data that was the input to the numerical tests, was modelled for a source absent of zero frequency. This is itself an important result since field data are always bandlimited. A more detailed analysis of the issues surrounding missing low frequency and the leading order imaging series is given in Part II (Shaw and Weglein, 2004).

## 5 Conclusion

A prestack formulation of a leading order imaging series for constant angles of incidence and a 1D medium has been derived and analyzed for several analytic and synthetic numerical experiments. The results illustrate the improvement in the predicted depths of the reflectors that are mislocated by conventional depth imaging (which corresponds to the first term in


Figure 9: Results of imaging the data in Fig. 8. On the left is the first term in the series: the result of an imaging algorithm that is linear in the data. On the right is the result of the leading order imaging series. The yellow lines are the actual depths of the reflectors.
the series). The effect of the leading order imaging series can be visualized as correcting the residual moveout of common image gathers that are imaged with the wrong velocity.

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## A Derivation of the first term, $\alpha_{1}$

The first term in the inverse series is a linear inversion of the scattered field. Beginning with (5), the data $\tilde{D}$ are related to $\alpha_{1}$ by

$$
\begin{equation*}
\tilde{D}\left(\vec{x}_{g} \mid \vec{x}_{s} ; \omega\right)=\int_{-\infty}^{\infty} d x^{\prime} \int_{-\infty}^{\infty} d y^{\prime} \int_{-\infty}^{\infty} d z^{\prime} G_{0}\left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(\vec{x}^{\prime}\right) G_{0}\left(\vec{x}^{\prime} \mid \vec{x}_{s} ; \omega\right) \tag{37}
\end{equation*}
$$

where $k_{0}=\omega / c_{0}$. The two reference Green's functions in (37) satisfy

$$
\begin{equation*}
\left(\nabla^{2}+\frac{\omega^{2}}{c_{0}^{2}}\right) G_{0}=-\delta \tag{38}
\end{equation*}
$$

and the causal solutions are (see, for example, DeSanto (1992))

$$
\begin{align*}
& G_{0}\left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right)=\frac{1}{(2 \pi)^{3}} \int_{-\infty}^{\infty} d k_{x^{\prime}} \int_{-\infty}^{\infty} d k_{y^{\prime}} \int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{x^{\prime}}\left(x_{g}-x^{\prime}\right)} e^{i k_{y^{\prime}}\left(y_{g}-y^{\prime}\right)} e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x^{\prime}}^{2}-k_{y^{\prime}}^{2}-k_{z^{\prime}}^{2}}  \tag{39}\\
& G_{0}\left(\vec{x}^{\prime} \mid \vec{x}_{s} ; \omega\right)=\frac{1}{(2 \pi)^{3}} \int_{-\infty}^{\infty} d k_{x_{s}} \int_{-\infty}^{\infty} d k_{y_{s}} \int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{x_{s}}\left(x^{\prime}-x_{s}\right)} e^{i k_{y_{s}}\left(y^{\prime}-y_{s}\right)} e^{i k_{z_{s}}\left(z^{\prime}-z_{s}\right)}}{k_{0}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2}-k_{z_{s}}^{2}} . \tag{40}
\end{align*}
$$

Implicit in (37) is that the incident field is the result of a point source and not a plane wave. Substituting these Green's functions into (37) yields

$$
\begin{align*}
& \tilde{D}\left(\vec{x}_{g} \mid \vec{x}_{s} ; \omega\right)= \frac{1}{(2 \pi)^{6}} \int_{-\infty}^{\infty} d x^{\prime} \int_{-\infty}^{\infty} d y^{\prime} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d k_{x^{\prime}} \int_{-\infty}^{\infty} d k_{y^{\prime}} \\
& \times \int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{x^{\prime}}\left(x_{g}-x^{\prime}\right)} e^{i k_{y^{\prime}}\left(y_{g}-y^{\prime}\right)} e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x^{\prime}}^{2}-k_{y^{\prime}}^{2}-k_{z^{\prime}}^{2}} k_{0}^{2} \alpha_{1}\left(\vec{x}^{\prime}\right) \\
& \times \int_{-\infty}^{\infty} d k_{x_{s}} \int_{-\infty}^{\infty} d k_{y_{s}} \int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{x_{s}}\left(x^{\prime}-x_{s}\right.} e^{i k_{y_{s}}\left(y^{\prime}-y_{s}\right)} e^{i k_{z_{s}}\left(z^{\prime}-z_{s}\right)}}{k_{0}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2}-k_{z_{s}}^{2}} . \tag{41}
\end{align*}
$$

Performing a double Fourier transform over $x_{g}$ and $y_{g}$

$$
\begin{align*}
& \tilde{D}\left(k_{x_{g}}, k_{y_{g}}, z_{g} \mid \vec{x}_{s} ; \omega\right)= \int_{-\infty}^{\infty} d x_{g} \int_{-\infty}^{\infty} d y_{g} \tilde{D}\left(x_{g}, y_{g}, z_{g} \mid \vec{x}_{s} ; \omega\right) e^{-i k_{x_{g}} x_{g}} e^{-i k_{y_{g} y_{g}}} \\
&= \frac{1}{(2 \pi)^{6}} \underbrace{\int_{-\infty}^{\infty} d x_{g} \int_{-\infty}^{\infty} d y_{g}} \int_{-\infty}^{\infty} d x^{\prime} \int_{-\infty}^{\infty} d y^{\prime} \int_{-\infty}^{\infty} d z^{\prime} \\
& \times \int_{-\infty}^{\infty} d k_{x^{\prime}} \int_{-\infty}^{\infty} d k_{y^{\prime}} \underbrace{i e^{i x_{g}\left(k_{x^{\prime}}-k_{x_{g} g}\right)} e^{i y_{g}\left(\left(k_{y^{\prime}}-k_{y_{g}}\right)\right.}} \\
& \times \int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{-i k_{x^{\prime}} x^{\prime}} e^{-i k_{y^{\prime} y^{\prime}}} e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x^{\prime}}^{2}-k_{y^{\prime}}^{2}-k_{z^{\prime}}^{2}} \\
& \times k_{0}^{2} \alpha_{1}\left(\vec{x}^{\prime}\right) \int_{-\infty}^{\infty} d k_{x_{s}} \int_{-\infty}^{\infty} d k_{y_{s}} \int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{x_{s}}\left(x^{\prime}-x_{s}\right)} e^{i k_{y_{s}}\left(y^{\prime}-y_{s}\right)} e^{i k_{z_{s}}\left(z^{\prime}-z_{s}\right)}}{k_{0}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2}-k_{z_{s}}^{2}} . \tag{42}
\end{align*}
$$

Carrying out the integrations over $x_{g}$ and $y_{g}$ (braced terms) produces two delta functions $(2 \pi)^{2} \delta\left(k_{x^{\prime}}-k_{x_{g}}\right) \delta\left(k_{y^{\prime}}-k_{y_{g}}\right)$. Then the integrations over $k_{x^{\prime}}$ and $k_{y^{\prime}}$ can be performed giving

$$
\begin{align*}
& \tilde{D}\left(k_{x_{g}}, k_{y_{g}}, z_{g} \mid \vec{x}_{s} ; \omega\right)= \frac{1}{(2 \pi)^{4}} \int_{-\infty}^{\infty} d x^{\prime} \int_{-\infty}^{\infty} d y^{\prime} \int_{-\infty}^{\infty} d z^{\prime} \\
& \times \int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{-i k_{x_{g} x^{\prime}}} e^{-i k_{y_{g}} y^{\prime}} e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime}}^{2}} k_{0}^{2} \alpha_{1}\left(\vec{x}^{\prime}\right) \\
& \times \int_{-\infty}^{\infty} d k_{x_{s}} \int_{-\infty}^{\infty} d k_{y_{s}} \int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{x_{s}}\left(x^{\prime}-x_{s}\right)} e^{i k_{y_{s}}\left(y^{\prime}-y_{s}\right)} e^{i k_{z_{s}}\left(z^{\prime}-z_{s}\right)}}{k_{0}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2}-k_{z_{s}}^{2}} \tag{43}
\end{align*}
$$

Now assume that the actual medium is invariant in the $x$ and $y$ directions, i.e.,

$$
\begin{equation*}
\alpha_{1}(\vec{x})=\alpha_{1}(z) . \tag{44}
\end{equation*}
$$

Collecting the exponentials in $x^{\prime}$ and $y^{\prime}$ and then carrying out the integrations over these variable produces two more delta functions (braced terms below) allowing integration over $k_{x_{s}}$ and $k_{y_{s}}$ :

$$
\begin{align*}
& \tilde{D}\left(k_{x_{g}}, k_{y_{g}}, z_{g} \mid \vec{x}_{s} ; \omega\right)= \frac{1}{(2 \pi)^{4}} \underbrace{\int_{-\infty}^{\infty} d x^{\prime} \int_{-\infty}^{\infty} d y^{\prime}}_{-\infty} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d k_{z^{\prime}} \\
& \times \frac{e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime}}^{2}} k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) \\
& \times \int_{-\infty}^{\infty} d k_{x_{s}} \underbrace{i x^{\prime}\left(k_{x_{s}}-k_{x_{g}}\right)} \\
& \times \int_{-\infty}^{\infty} d k_{z_{y_{s}}}^{\infty} \frac{e^{-i k_{x_{s}} x_{s}} e^{-i k_{y_{s}} y_{s}} e^{i k_{z_{s}}\left(z^{\prime}-z_{s}\right)}}{k_{0}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2}-k_{z_{s}}^{2}} \\
&= \frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d k_{z^{\prime}}-k_{\left.y_{g}\right)} \\
& e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}  \tag{45}\\
& \times k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d k_{z_{s}}^{2}-k_{y_{s}}^{2}-k_{z^{\prime}}^{2} \\
& e^{-i k_{x_{g} x_{s}}} e^{-i k_{y_{g} y_{s}}} e^{i k_{z_{s}}\left(z^{\prime}-z_{s}\right)} \\
& k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z_{s}}^{2}
\end{align*} .
$$

Note that the integrations over $x^{\prime}$ and $y^{\prime}$ in (45), for a laterally invariant medium where $\alpha_{1}$ is not a function of $x^{\prime}$ or $y^{\prime}$, demonstrates that $k_{x_{g}}=k_{x_{s}}$ and $k_{y_{g}}=k_{y_{s}}$. Define the vertical wavenumbers

$$
\begin{equation*}
q_{g}^{2}=k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2} \tag{46}
\end{equation*}
$$

and

$$
\begin{equation*}
q_{s}^{2}=k_{0}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2} \tag{47}
\end{equation*}
$$

which, for the case where $\alpha$ is only a function of $z$, are equal $\left(q_{g}=q_{s}\right)$. Substituting (46) into (45),

$$
\begin{align*}
\tilde{D}\left(k_{x_{g}}, k_{y_{g}}, z_{g} \mid \vec{x}_{s} ; \omega\right)= & \frac{1}{(2 \pi)^{2}} e^{-i k_{x_{g}} x_{s}} e^{-i k_{y_{g}} y_{s}} \int_{-\infty}^{\infty} d z^{\prime} k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) \\
& \times \int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{q_{g}^{2}-k_{z^{\prime}}^{2}} \int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{z_{s}}\left(z^{\prime}-z_{s}\right)}}{q_{g}^{2}-k_{z_{s}}^{2}} . \tag{48}
\end{align*}
$$

We are now in a position to perform the integrals with respect to $k_{z^{\prime}}$ and $k_{z_{s}}$ (see, e.g., DeSanto (1992), page 57):

$$
\begin{align*}
\int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{q_{g}^{2}-k_{z^{\prime}}^{2}} & =-\int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{\left(k_{z^{\prime}}-q_{g}\right)\left(k_{z^{\prime}}+q_{g}\right)} \\
& =-\frac{\pi i}{q_{g}} e^{i q_{g}\left|z_{g}-z^{\prime}\right|} \tag{49}
\end{align*}
$$

and similarly

$$
\begin{equation*}
\int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{z_{s}}\left(z^{\prime}-z_{s}\right)}}{q_{g}^{2}-k_{z_{s}}^{2}}=-\frac{\pi i}{q_{g}} e^{i q_{g}\left|z^{\prime}-z_{s}\right|} \tag{50}
\end{equation*}
$$

Substituting (49) and (50) into (48) gives

$$
\begin{align*}
& \tilde{D}\left(k_{x_{g}}, k_{y_{g}}, z_{g} \mid \vec{x}_{s} ; \omega\right)=\frac{1}{(2 \pi)^{2}} e^{-i k_{x_{g}} x_{s}} e^{-i k_{y_{g}} y_{s}} \int_{-\infty}^{\infty} d z^{\prime} k_{0}^{2} \alpha_{1}\left(z^{\prime}\right)\left(-\frac{\pi i}{q_{g}} e^{i q_{g}\left|z_{g}-z^{\prime}\right|}\right)\left(-\frac{\pi i}{q_{g}} e^{i q_{g}\left|z^{\prime}-z_{s}\right|}\right) \\
&=\frac{e^{-i k_{x_{g}} x_{s}}}{-i k_{y_{g} y_{s}}}-4 q_{g}^{2} \\
&-\infty  \tag{51}\\
&-\infty z^{\prime} e^{i q_{g}\left(z^{\prime}-z_{g}\right)} k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) e^{i q_{g}\left(z^{\prime}-z_{s}\right)} \\
&=\frac{k_{0}^{2}}{-4 q_{g}^{2}} e^{-i k_{x_{g} x_{s}}} e^{-i k_{y_{g}} y_{s}} e^{-i q_{g}\left(z_{g}+z_{s}\right)} \tilde{\alpha}_{1}\left(-2 q_{g}\right)
\end{align*}
$$

where in (51) we have assumed that the scattering points are below the measurement surface $\left(z^{\prime}>z_{g}\right.$ and $\left.z^{\prime}>z_{s}\right)$. Performing a double inverse Fourier transform

$$
\begin{align*}
\tilde{D}\left(\vec{x}_{g} \mid \vec{x}_{s} ; \omega\right) & =\frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} d k_{x_{g}} e^{i k_{x_{g}} x_{g}} \int_{-\infty}^{\infty} d k_{y_{g}} e^{i k_{y_{g}} y_{g}} \tilde{D}\left(k_{x_{g}}, k_{y_{g}}, z_{g} \mid \vec{x}_{s} ; \omega\right) \\
& =\frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} d k_{x_{g}} e^{i k_{x_{g} x_{g}}} \int_{-\infty}^{\infty} d k_{y_{g}} e^{i k_{y_{g} y_{g}}} \frac{k_{0}^{2}}{-4 q_{g}^{2}} e^{-i k_{x_{g} x_{s}}} e^{-i k_{y_{g}} y_{s}} e^{-i q_{g}\left(z_{g}+z_{s}\right)} \tilde{\alpha}_{1}\left(-2 q_{g}\right) \\
& =\frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} d k_{x_{g}} \int_{-\infty}^{\infty} d k_{y_{g}} \frac{k_{0}^{2}}{-4 q_{g}^{2}} \tilde{\alpha}_{1}\left(-2 q_{g}\right) e^{i k_{x_{g}}\left(x_{g}-x_{s}\right)} e^{i k_{y_{g}}\left(y_{g}-y_{s}\right)} e^{-i q_{g}\left(z_{g}+z_{s}\right)} . \tag{52}
\end{align*}
$$

We proceed by changing from cartesian to cylindrical coordinates where

$$
\begin{equation*}
\int_{-\infty}^{\infty} d k_{x_{g}} \int_{-\infty}^{\infty} d k_{y_{g}}=\int_{0}^{\infty} k_{r} d k_{r} \int_{0}^{2 \pi} d \tilde{\phi} \tag{53}
\end{equation*}
$$

Substituting (53) into (52) yields

$$
\begin{equation*}
\tilde{D}(r, z ; \omega)=\frac{1}{(2 \pi)^{2}} \int_{0}^{\infty} d k_{r} \int_{0}^{2 \pi} d \tilde{\phi} \frac{k_{0}^{2} k_{r}}{-4 q_{g}^{2}} \tilde{\alpha}_{1}\left(-2 q_{g}\right) e^{i k_{r} \cos \tilde{\phi} r \cos \phi} e^{i k_{r} \sin \tilde{\phi} r \sin \phi} e^{-i q_{g}\left(z_{g}+z_{s}\right)} \tag{54}
\end{equation*}
$$

and, since,

$$
\begin{equation*}
\int_{0}^{2 \pi} e^{i k_{r} r(\cos \tilde{\phi} \cos \phi+\sin \tilde{\phi} \sin \phi)} d \tilde{\phi}=\int_{0}^{2 \pi} e^{i k_{r} r(\cos (\tilde{\phi}-\phi))} d \tilde{\phi}=2 \pi J_{0}\left(k_{r} r\right), \tag{55}
\end{equation*}
$$

then (54) becomes

$$
\begin{equation*}
\tilde{D}(r ; \omega)=\frac{1}{(2 \pi)} \int_{0}^{\infty} \frac{k_{0}^{2}}{-4 q_{g}^{2}} \tilde{\alpha}_{1}\left(-2 q_{g}\right) e^{-i q_{g}\left(z_{g}+z_{s}\right)} J_{0}\left(k_{r} r\right) k_{r} d k_{r} \tag{56}
\end{equation*}
$$

which is an expression for the scattered field in terms of $\tilde{\alpha}_{1}$. Equation (56) can be inverted by recognizing the Fourier-Bessel transform pairs

$$
\begin{align*}
g(r) & =\int_{0}^{\infty} G\left(k_{r}\right) J_{0}\left(k_{r} r\right) k_{r} d k_{r}  \tag{57}\\
G\left(k_{r}\right) & =\int_{0}^{\infty} g(r) J_{0}\left(k_{r} r\right) r d r \tag{58}
\end{align*}
$$

and leads to

$$
\begin{equation*}
\frac{1}{(2 \pi)} \frac{k_{0}^{2}}{-4 q_{g}^{2}} \tilde{\alpha}_{1}\left(-2 q_{g}\right) e^{-i q_{g}\left(z_{g}+z_{s}\right)}=\int_{0}^{\infty} \tilde{D}(r ; \omega) J_{0}\left(k_{r} r\right) r d r . \tag{59}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\tilde{\alpha}_{1}\left(-2 q_{g}\right)=2 \pi \frac{-4 q_{g}^{2}}{k_{0}^{2}} e^{i q_{g}\left(z_{g}+z_{s}\right)} \int_{0}^{\infty} \tilde{D}(r ; \omega) J_{0}\left(k_{r} r\right) r d r . \tag{60}
\end{equation*}
$$

where

$$
\begin{equation*}
q_{g}=\frac{\omega}{c_{0}} \sqrt{1-\frac{k_{r}^{2} c_{0}^{2}}{\omega^{2}}} . \tag{61}
\end{equation*}
$$

From (60), we see that $\tilde{\alpha}_{1}$ is over-determined (there are more free variables on the right-hand side than on the left). Inverse Fourier transforming both sides of (60) gives

$$
\begin{align*}
\alpha_{1}(z) & =\frac{2}{2 \pi} \int_{-\infty}^{\infty} \tilde{\alpha}_{1}\left(-2 q_{g}\right) e^{-2 i q_{g} z} d q_{g} \\
& =-8 \int_{-\infty}^{\infty} \frac{q_{g}^{2}}{k_{0}^{2}} e^{-i q_{g}\left(2 z-\left(z_{g}+z_{s}\right)\right)} \int_{0}^{\infty} \tilde{D}(r ; \omega) J_{0}\left(k_{r} r\right) r d r d q_{g} \tag{62}
\end{align*}
$$

Considering fixed angles of incidence, $\theta_{0}$, leads to a number of different estimates of $\alpha_{1}$, denoted by $\alpha_{1}\left(z, \theta_{0}\right)$. Fixing $\theta_{0}$ is the same as fixing horizontal and vertical slownesses, $p=p_{0}$ and $\zeta=\zeta_{0}$, respectively, where

$$
p_{0}=\frac{\sin \theta_{0}}{c_{0}} \text { and } \zeta_{0}=\frac{\cos \theta_{0}}{c_{0}} .
$$

However, $q_{g}$ is still allowed to vary through the variation in $\omega$ (since $q_{g}=\omega \zeta_{0}$ ). We proceed by changing variables from $q_{g}$ to $\omega$ :

$$
\begin{equation*}
\alpha_{1}\left(z, \theta_{0}\right)=-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{-\infty}^{\infty} e^{-i \omega \zeta_{0}\left(2 z-\left(z_{g}+z_{s}\right)\right)} \int_{0}^{\infty} \tilde{D}(r ; \omega) J_{0}\left(\omega p_{0} r\right) r d r d \omega \tag{63}
\end{equation*}
$$

Define $\tau_{0}=\zeta_{0}\left(2 z-\left(z_{g}+z_{s}\right)\right)$ and substitute (a) the temporal Fourier transform of the data $D(r, t)$ for $\tilde{D}(r ; \omega)$ and (b) the integral form of the Bessel function gives

$$
\begin{align*}
\alpha_{1}\left(z, \theta_{0}\right) & =-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{-\infty}^{\infty} d \omega e^{-i \omega \tau_{0}} \int_{0}^{\infty} r d r\left(\int_{-\infty}^{\infty} D(r ; t) e^{i \omega t} d t\right)\left(\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{i \omega p_{0} r \cos \phi} d \phi\right) \\
& =-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{-\infty}^{\infty} d \omega \int_{0}^{\infty} r d r\left(\int_{-\infty}^{\infty} D(r ; t) e^{i \omega\left(t-\left(\tau_{0}-p_{0} r \cos \phi\right)\right)} d t\right)\left(\frac{1}{2 \pi} \int_{0}^{2 \pi} d \phi\right) \\
& =-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{0}^{\infty} r d r\left(\int_{-\infty}^{\infty} D(r ; t) 2 \pi \delta\left(t-\left(\tau_{0}-p_{0} r \cos \phi\right)\right) d t\right) \\
& =-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{0}^{2 \pi} \int_{0}^{\infty} D\left(r ; \tau_{0}-p_{0} r \cos \phi\right) r d r d \phi \tag{64}
\end{align*}
$$

Changing back to cartesian coordinates, where

$$
r=\sqrt{x^{2}+y^{2}}, \quad \phi=\arccos \left(\frac{x}{\sqrt{x^{2}+y^{2}}}\right), \quad \phi=\arcsin \left(\frac{y}{\sqrt{x^{2}+y^{2}}}\right)
$$

and the partial derivatives are

$$
\begin{aligned}
\frac{\partial r}{\partial x} & =\frac{x}{\sqrt{x^{2}+y^{2}}}=\frac{x}{r} \\
\frac{\partial r}{\partial y} & =\frac{y}{\sqrt{x^{2}+y^{2}}}=\frac{y}{r} \\
\frac{\partial \phi}{\partial x} & =\frac{-1}{\sqrt{1-\left(\frac{x}{\sqrt{x^{2}+y^{2}}}\right)^{2}}} \times \frac{\left(\sqrt{x^{2}+y^{2}}-\frac{x^{2}}{\sqrt{x^{2}+y^{2}}}\right)}{\left(x^{2}+y^{2}\right)} \\
& =\frac{\left(\frac{x^{2}}{r}-r\right)}{r^{2} \sqrt{1-\left(\frac{x^{2}}{r^{2}}\right)}}=\frac{\frac{1}{r}\left(x^{2}-r^{2}\right)}{r \sqrt{r^{2}-x^{2}}}=\frac{-y}{r^{2}}
\end{aligned}
$$

$$
\frac{\partial \phi}{\partial y}=\frac{1}{\sqrt{1-\left(\frac{y}{\sqrt{x^{2}+y^{2}}}\right)^{2}}} \times \frac{\left(\sqrt{x^{2}+y^{2}}-\frac{y^{2}}{\sqrt{x^{2}+y^{2}}}\right)}{\left(x^{2}+y^{2}\right)}
$$

$$
=\frac{\left(r-\frac{y^{2}}{r}\right)}{r^{2} \sqrt{1-\left(\frac{y^{2}}{r^{2}}\right)}}=\frac{\frac{1}{r}\left(r^{2}-y^{2}\right)}{r \sqrt{r^{2}-y^{2}}}=\frac{x}{r^{2}} .
$$

So the Jacobian is

$$
\frac{\partial r}{\partial x} \frac{\partial \phi}{\partial y}-\frac{\partial r}{\partial y} \frac{\partial \phi}{\partial x}=\frac{x^{2}}{r^{3}}+\frac{y^{2}}{r^{3}}=\frac{1}{r}
$$

and therefore

$$
\begin{equation*}
\alpha_{1}\left(z, \theta_{0}\right)=-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D\left(x, y ; \tau_{0}-x p_{0}\right) d x d y \tag{65}
\end{equation*}
$$

where $\tau_{0}=\zeta_{0}\left(2 z-\left(z_{g}+z_{s}\right)\right)$ and $p_{0}=\sin \theta_{0} / c_{0}$. Equation (65) is recognizable as the slant stack of the recorded data (Treitel et al., 1982).

## B Derivation of the second term, $\alpha_{2}$

The integral equation for the second term in the inverse series for this acoustic problem is

$$
\begin{align*}
\int_{-\infty}^{\infty} G_{0}\left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right) & k_{0}^{2} \alpha_{2}\left(\vec{x}^{\prime}\right) G_{0}\left(\vec{x}^{\prime} \mid \vec{x}_{s} ; \omega\right) d \vec{x}^{\prime} \\
= & -\int_{-\infty}^{\infty} d \vec{x}^{\prime} G_{0}\left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(\vec{x}^{\prime}\right) \\
& \times \int_{-\infty}^{\infty} d \vec{x}^{\prime \prime} G_{0}\left(\vec{x}^{\prime} \mid \vec{x}^{\prime \prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(\vec{x}^{\prime \prime}\right) G_{0}\left(\vec{x}^{\prime \prime} \mid \vec{x}_{s} ; \omega\right) \tag{66}
\end{align*}
$$

Upon substitution of the causal Green's functions, the left-hand side becomes

$$
\begin{align*}
L H S= & \frac{1}{(2 \pi)^{6}} \int_{-\infty}^{\infty} d x^{\prime} \int_{-\infty}^{\infty} d y^{\prime} \int_{-\infty}^{\infty} d z^{\prime} \\
& \times \int_{-\infty}^{\infty} d k_{x^{\prime}} \int_{-\infty}^{\infty} d k_{y^{\prime}} \int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{x^{\prime}}\left(x_{g}-x^{\prime}\right)} e^{i k_{y^{\prime}}\left(y_{g}-y^{\prime}\right)} e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x^{\prime}}^{2}-k_{y^{\prime}}^{2}-k_{z^{\prime}}^{2}} \\
& \times k_{0}^{2} \alpha_{2}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d k_{x_{s}} \int_{-\infty}^{\infty} d k_{y_{s}} \int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{x_{s}}\left(x^{\prime}-x_{s}\right)} e^{i k_{y_{s}}\left(y^{\prime}-y_{s}\right)} e^{i k_{z_{s}}\left(z^{\prime}-z_{s}\right)}}{k_{0}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2}-k_{z_{s}}^{2}} \tag{67}
\end{align*}
$$

and the right-hand side is

$$
\begin{align*}
R H S= & \frac{-1}{(2 \pi)^{9}} \int_{-\infty}^{\infty} d x^{\prime} \int_{-\infty}^{\infty} d y^{\prime} \int_{-\infty}^{\infty} d z^{\prime} \\
& \times \int_{-\infty}^{\infty} d k_{x^{\prime}} \int_{-\infty}^{\infty} d k_{y^{\prime}} \int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{x^{\prime}}\left(x_{g}-x^{\prime}\right)} e^{i k_{y^{\prime}}\left(y_{g}-y^{\prime}\right)} e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x^{\prime}}^{2}-k_{y^{\prime}}^{2}-k_{z^{\prime}}^{2}} \\
& \times k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d x^{\prime \prime} \int_{-\infty}^{\infty} d y^{\prime \prime} \int_{-\infty}^{\infty} d z^{\prime \prime} \\
& \times \int_{-\infty}^{\infty} d k_{x^{\prime \prime}} \int_{-\infty}^{\infty} d k_{y^{\prime \prime}} \int_{-\infty}^{\infty} d k_{z^{\prime \prime}} \frac{e^{i k_{x^{\prime \prime}}\left(x^{\prime}-x^{\prime \prime}\right)} e^{i k_{y^{\prime \prime}}\left(y^{\prime}-y^{\prime \prime}\right)} e^{i k_{z^{\prime \prime}}\left(z^{\prime}-z^{\prime \prime}\right)}}{k_{0}^{2}-k_{x^{\prime \prime}}^{2}-k_{y^{\prime \prime}}^{2}-k_{z^{\prime \prime}}^{2}} k_{0}^{2} \alpha_{1}\left(z^{\prime \prime}\right) \\
& \times \int_{-\infty}^{\infty} d k_{x_{s}} \int_{-\infty}^{\infty} d k_{y_{s}} \int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{x_{s}}\left(x^{\prime \prime}-x_{s}\right)} e^{i k_{y_{s}}\left(y^{\prime \prime}-y_{s}\right)} e^{i k_{z_{s}}\left(z^{\prime \prime}-z_{s}\right)}}{k_{0}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2}-k_{z_{s}}^{2}} . \tag{68}
\end{align*}
$$

As with deriving an equation for $\alpha_{1}$, we Fourier transform both sides over $x_{g}$ and $y_{g}$ and perform the integrations over $k_{x^{\prime}}$ and $k_{y^{\prime}}$. Therefore,

$$
\begin{align*}
& L H S \rightarrow \frac{1}{(2 \pi)^{4}} \\
& \int_{-\infty}^{\infty} d x^{\prime} \int_{-\infty}^{\infty} d y^{\prime} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{-i k_{x_{g}} x^{\prime}} e^{-i k_{y_{g} y^{\prime}}} e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime}}^{2}}  \tag{69}\\
& \times k_{0}^{2} \alpha_{2}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d k_{x_{s}} \int_{-\infty}^{\infty} d k_{y_{s}} \int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{x_{s}}\left(x^{\prime}-x_{s}\right.} e^{i k_{y_{s}}\left(y^{\prime}-y_{s}\right)} e^{i k_{z_{s}}\left(z^{\prime}-z_{s}\right)}}{k_{0}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2}-k_{z_{s}}^{2}} \\
& R H S \rightarrow \frac{-1}{(2 \pi)^{7}} \int_{-\infty}^{\infty} d x^{\prime} \int_{-\infty}^{\infty} d y^{\prime} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d k_{z^{\prime}} \\
& \times \frac{e^{-i k_{x_{g} x^{\prime}} e^{-i k_{y_{g} y^{\prime}} e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}} k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime}}^{2}}{k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d x^{\prime \prime} \int_{-\infty}^{\infty} d y^{\prime \prime} \int_{-\infty}^{\infty} d z^{\prime \prime}} \\
& \times \int_{-\infty}^{\infty} d k_{x^{\prime \prime}} \int_{-\infty}^{\infty} d k_{y^{\prime \prime}} \int_{-\infty}^{\infty} d k_{z^{\prime \prime}} \\
& \times \frac{e^{i k_{x^{\prime \prime}}\left(x^{\prime}-x^{\prime \prime}\right)} e^{i k_{y^{\prime \prime}}\left(y^{\prime}-y^{\prime \prime}\right)} e^{i k_{z^{\prime \prime}}\left(z^{\prime}-z^{\prime \prime}\right)}}{k_{0}^{2}-k_{x^{\prime \prime}}^{2}-k_{y^{\prime \prime}}^{2}-k_{z^{\prime \prime}}^{2} \alpha_{1}\left(z^{\prime \prime}\right)}  \tag{70}\\
& \times \int_{-\infty}^{\infty} d k_{x_{s}} \int_{-\infty}^{\infty} d k_{y_{s}} \int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{x_{s}}\left(x^{\prime \prime}-x_{s}\right)} e^{i k_{y_{s}\left(y^{\prime \prime}-y_{s}\right)}} e^{i k_{z_{s}\left(z^{\prime \prime}-z_{s}\right)}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2}-k_{z_{s}}^{2}} .}{} .
\end{align*}
$$

Collecting the exponentials in $x^{\prime}$ and $y^{\prime}$ and performing the integrations with respect to these variables produces delta functions $2 \pi \delta\left(k_{x_{s}}-k_{x_{g}}\right)$ and $2 \pi \delta\left(k_{y_{s}}-k_{y_{g}}\right)(L H S)$ and $2 \pi \delta\left(k_{x^{\prime \prime}}-k_{x_{g}}\right)$ and $2 \pi \delta\left(k_{y^{\prime \prime}}-k_{y_{g}}\right)(R H S)$ allowing for the $k_{x_{s}}$ and $k_{y_{s}}(L H S)$ and $k_{x^{\prime \prime}}$ and $k_{y^{\prime \prime}}(R H S)$ integrals, respectively, to be carried out:

$$
\begin{align*}
L H S & \frac{1}{(2 \pi)^{4}} \underbrace{\int_{-\infty}^{\infty} d x^{\prime}} \underbrace{\int_{-\infty}^{\infty} d y^{\prime}} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime}}^{2}} \\
& \times k_{0}^{2} \alpha_{2}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d k_{x_{s}} \int_{-\infty}^{\infty} d k_{y_{s}} \int_{-\infty}^{\infty} d k_{z_{s}} \underbrace{e^{i\left(k_{x_{s}}-k_{x_{g}}\right) x^{\prime}}} \underbrace{i\left(k_{y_{s}}-k_{y_{g}}\right) y^{\prime}} \\
& \times \frac{e^{-i k_{x_{s} x_{s}}} e^{-i k_{y_{s}} y_{s}} e^{i k_{z_{s}}\left(z^{\prime}-z_{s}\right)}}{k_{0}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2}-k_{z_{s}}^{2}} \\
\rightarrow & \frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime}}^{2}} k_{0}^{2} \alpha_{2}\left(z^{\prime}\right) \\
& \times \int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{-i k_{x_{g}} x_{s}} e^{-i k_{y_{g}} y_{s}} e^{i k_{z_{s}}\left(z^{\prime}-z_{s}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z_{s}}^{2}} \tag{71}
\end{align*}
$$

$$
\begin{align*}
& R H S \rightarrow \frac{-1}{(2 \pi)^{7}} \underbrace{\int_{-\infty}^{\infty} d x^{\prime}} \underbrace{\int_{-\infty}^{\infty} d y^{\prime}} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime}}^{2}} \\
& \times k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d x^{\prime \prime} \int_{-\infty}^{\infty} d y^{\prime \prime} \int_{-\infty}^{\infty} d z^{\prime \prime} \int_{-\infty}^{\infty} d k_{x^{\prime \prime}} \int_{-\infty}^{\infty} d k_{y^{\prime \prime}} \int_{-\infty}^{\infty} d k_{z^{\prime \prime}} \\
& \times \underbrace{e^{i\left(k_{x^{\prime \prime}}-k_{x_{g}}\right) x^{\prime}}} \underbrace{e^{i\left(k_{y^{\prime \prime}}-k_{y_{g}}\right) y^{\prime}}} \frac{e^{-i k_{x^{\prime \prime}} x^{\prime \prime}} e^{-i k_{y^{\prime \prime}} y^{\prime \prime}} e^{i k_{z^{\prime \prime}}\left(z^{\prime}-z^{\prime \prime}\right)}}{k_{0}^{2}-k_{x^{\prime \prime}}^{2}-k_{y^{\prime \prime}}^{2}-k_{z^{\prime \prime}}^{2}} k_{0}^{2} \alpha_{1}\left(z^{\prime \prime}\right) \\
& \times \int_{-\infty}^{\infty} d k_{x_{s}} \int_{-\infty}^{\infty} d k_{y_{s}} \int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{x_{s}}\left(x^{\prime \prime}-x_{s}\right)} e^{i k_{y_{s}}\left(y^{\prime \prime}-y_{s}\right)} e^{i k_{z_{s}}\left(z^{\prime \prime}-z_{s}\right)}}{k_{0}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2}-k_{z_{s}}^{2}}  \tag{72}\\
& \rightarrow \frac{-1}{(2 \pi)^{5}} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime}}^{2}} k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) \\
& \times \int_{-\infty}^{\infty} d x^{\prime \prime} \int_{-\infty}^{\infty} d y^{\prime \prime} \int_{-\infty}^{\infty} d z^{\prime \prime} \int_{-\infty}^{\infty} d k_{z^{\prime \prime}} \\
& \times \frac{e^{-i k_{x_{g}} x^{\prime \prime}} e^{-i k_{y_{g}} y^{\prime \prime}} e^{i k_{z^{\prime \prime}}\left(z^{\prime}-z^{\prime \prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime \prime}}^{2}} k_{0}^{2} \alpha_{1}\left(z^{\prime \prime}\right) \\
& \times \int_{-\infty}^{\infty} d k_{x_{s}} \int_{-\infty}^{\infty} d k_{y_{s}} \int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{x_{s}}\left(x^{\prime \prime}-x_{s}\right)} e^{i k_{y_{s}}\left(y^{\prime \prime}-y_{s}\right)} e^{i k_{z_{s}}\left(z^{\prime \prime}-z_{s}\right)}}{k_{0}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2}-k_{z_{s}}^{2}} \tag{73}
\end{align*}
$$

As in the inversion for $\alpha_{1}$, the two wavenumber integrals (over $k_{z^{\prime}}$ and $k_{z_{s}}$ ) on the left-hand side can now be evaluated:

$$
\begin{align*}
\text { LHS } \rightarrow & \frac{k_{0}^{2}}{(2 \pi)^{2}} e^{-i k_{x_{g}} x_{s}} e^{-i k_{y_{g}} y_{s}} \int_{-\infty}^{\infty} d z^{\prime} \alpha_{2}\left(z^{\prime}\right)\left(-\int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{\left(k_{z^{\prime}}-q_{g}\right)\left(k_{z^{\prime}}+q_{g}\right)}\right) \\
& \times\left(-\int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{z_{s}}\left(z^{\prime}-z_{s}\right)}}{\left(k_{z_{s}}-q_{g}\right)\left(k_{z_{s}}+q_{g}\right)}\right) \\
\rightarrow & \frac{k_{0}^{2}}{(2 \pi)^{2}} e^{-i k_{x_{g} x_{s}}} e^{-i k_{y_{g} y_{s}}} \int_{-\infty}^{\infty} d z^{\prime} \alpha_{2}\left(z^{\prime}\right)\left(\frac{\pi i}{q_{g}} e^{i q_{g}\left|z_{g}-z^{\prime}\right|}\right)\left(\frac{\pi i}{q_{g}} e^{i q_{g}\left|z^{\prime}-z_{s}\right|}\right) \\
\rightarrow & \frac{-k_{0}^{2}}{4 q_{g}^{2}} e^{-i k_{x_{g} x_{s}}} e^{-i k_{y_{g} y_{s}}} e^{-i q_{g}\left(z_{g}+z_{s}\right)} \int_{-\infty}^{\infty} d z^{\prime} \alpha_{2}\left(z^{\prime}\right) e^{2 i q_{g} z^{\prime}} . \tag{74}
\end{align*}
$$

Meanwhile, the right-hand side can be simplified:

$$
\begin{align*}
& R H S \rightarrow \frac{-1}{(2 \pi)^{5}} \int_{-\infty}^{\infty} d z^{\prime} \underbrace{\int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime}}^{2}}} k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d x^{\prime \prime} e^{-i k_{x_{g} x^{\prime \prime}}} \\
& \times \int_{-\infty}^{\infty} d y^{\prime \prime} e^{-i k_{y_{g}} y^{\prime \prime}} \int_{-\infty}^{\infty} d z^{\prime \prime} \underbrace{\int_{-\infty}^{\infty} d k_{z^{\prime \prime}} \frac{e^{i k_{z^{\prime \prime}}\left(z^{\prime}-z^{\prime \prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime \prime}}^{2}}} k_{0}^{2} \alpha_{1}\left(z^{\prime \prime}\right) \\
& \times \int_{-\infty}^{\infty} d k_{x_{s}} e^{i k_{x_{s}}\left(x^{\prime \prime}-x_{s}\right)} \int_{-\infty}^{\infty} d k_{y_{s}} e^{i k_{y_{s}}\left(y^{\prime \prime}-y_{s}\right)} \underbrace{\int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{z_{s}}\left(z^{\prime \prime}-z_{s}\right)}}{k_{0}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2}-k_{z_{s}}^{2}}} \tag{75}
\end{align*}
$$

where

$$
\begin{align*}
& \int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime}}^{2}}=-\frac{\pi i}{q_{g}} e^{i q_{g}\left|z_{g}-z^{\prime}\right|}  \tag{76}\\
& \int_{-\infty}^{\infty} d k_{z^{\prime \prime}} \frac{e^{i k_{z^{\prime \prime}}\left(z^{\prime}-z^{\prime \prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime \prime}}^{2}}=-\frac{\pi i}{q_{g}} e^{i q_{g}\left|z^{\prime}-z^{\prime \prime}\right|}  \tag{77}\\
& \int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{z_{s}\left(z^{\prime \prime}-z_{s}\right)}^{2}} k_{0}^{2}-k_{x_{s}}^{2}-k_{y_{s}}^{2}-k_{z_{s}}^{2}}{}=-\frac{\pi i}{q_{s}} e^{i q_{s}\left|z^{\prime \prime}-z_{s}\right|} . \tag{78}
\end{align*}
$$

Substituting (76)-(78) into (75) yields

$$
\begin{align*}
R H S \rightarrow & \frac{-1}{(2 \pi)^{5}} \int_{-\infty}^{\infty} d z^{\prime}\left(-\frac{\pi i}{q_{g}} e^{i q_{g}\left(z^{\prime}-z_{g}\right)}\right) k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d x^{\prime \prime} e^{-i k_{x_{g}} x^{\prime \prime}} \\
& \times \int_{-\infty}^{\infty} d y^{\prime \prime} e^{-i k_{y_{g} y^{\prime \prime}}} \int_{-\infty}^{\infty} d z^{\prime \prime}\left(-\frac{\pi i}{q_{g}} e^{i q_{g}\left|z^{\prime}-z^{\prime \prime}\right|}\right) k_{0}^{2} \alpha_{1}\left(z^{\prime \prime}\right) \\
& \times \int_{-\infty}^{\infty} d k_{x_{s}} e^{i k_{x_{s}}\left(x^{\prime \prime}-x_{s}\right)} \int_{-\infty}^{\infty} d k_{y_{s}} e^{i k_{y_{s}}\left(y^{\prime \prime}-y_{s}\right)}\left(-\frac{\pi i}{q_{s}} e^{i q_{s}\left(z^{\prime \prime}-z_{s}\right)}\right) \tag{79}
\end{align*}
$$

and the integrations over $x^{\prime \prime}$ and $y^{\prime \prime}$ produce two more delta functions $2 \pi \delta\left(k_{x_{s}}-k_{x_{g}}\right)$ and $2 \pi \delta\left(k_{y_{s}}-k_{y_{g}}\right)$

$$
\begin{align*}
R H S \rightarrow & \frac{-1}{(2 \pi)^{3}} \int_{-\infty}^{\infty} d z^{\prime}\left(-\frac{\pi i}{q_{g}} e^{i q_{g}\left(z^{\prime}-z_{g}\right)}\right) k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d z^{\prime \prime}\left(-\frac{\pi i}{q_{g}} e^{i q_{g}\left|z^{\prime}-z^{\prime \prime}\right|}\right) k_{0}^{2} \alpha_{1}\left(z^{\prime \prime}\right) \\
& \times e^{-i k_{x_{g} x_{s}}} e^{-i k_{y_{g} y_{s}}}\left(-\frac{\pi i}{q_{s}} e^{i q_{g}\left(z^{\prime \prime}-z_{s}\right)}\right) . \tag{80}
\end{align*}
$$

Then expanding the absolute value in the exponential and simplifying:

$$
\begin{align*}
R H S \rightarrow & \frac{-i k_{0}^{4}}{8 q_{g}^{3}} \int_{-\infty}^{\infty} d z^{\prime} e^{i q_{g}\left(z^{\prime}-z_{g}\right)} \alpha_{1}\left(z^{\prime}\right)\left(\int_{-\infty}^{\infty} d z^{\prime \prime} H\left(z^{\prime}-z^{\prime \prime}\right) e^{i q_{g}\left(z^{\prime}-z^{\prime \prime}\right)} \alpha_{1}\left(z^{\prime \prime}\right)\right. \\
& \left.+\int_{-\infty}^{\infty} d z^{\prime \prime} H\left(z^{\prime \prime}-z^{\prime}\right) e^{i q_{g}\left(z^{\prime \prime}-z^{\prime}\right)} \alpha_{1}\left(z^{\prime \prime}\right)\right) e^{-i k_{x_{g} x_{s}}} e^{-i k_{y_{g} y_{s}} e^{i q_{g}\left(z^{\prime \prime}-z_{s}\right)}} \begin{aligned}
\rightarrow & \frac{-i k_{0}^{4}}{8 q_{g}^{3}} e^{-i k_{x_{g} x_{s}}} e^{-i k_{y_{g}} y_{s}} e^{-i q_{g}\left(z_{g}+z_{s}\right)} \int_{-\infty}^{\infty} d z^{\prime} \alpha_{1}\left(z^{\prime}\right) \\
& \times\left(2 \int_{-\infty}^{\infty} d z^{\prime \prime} H\left(z^{\prime}-z^{\prime \prime}\right) e^{i q_{g}\left(z^{\prime}-z^{\prime \prime}\right)} \alpha_{1}\left(z^{\prime \prime}\right)\right) e^{i q_{g}\left(z^{\prime}+z^{\prime \prime}\right)} \\
\rightarrow & \frac{-i k_{0}^{4}}{4 q_{g}^{3}} e^{-i k_{x_{g} x_{s}}} e^{-i k_{y_{g}} y_{s}} e^{-i q_{g}\left(z_{g}+z_{s}\right)} \int_{-\infty}^{\infty} d z^{\prime} \alpha_{1}\left(z^{\prime}\right) e^{2 i q_{g} z^{\prime}} \\
& \times \int_{-\infty}^{\infty} d z^{\prime \prime} H\left(z^{\prime}-z^{\prime \prime}\right) \alpha_{1}\left(z^{\prime \prime}\right) .
\end{aligned}
\end{align*}
$$

Now equating the left-hand and right-hand sides, (74) and (81), the phases $\exp \left(-i k_{x_{g}} x_{s}\right)$, $\exp \left(-i k_{y_{g}} y_{s}\right)$ and $\exp \left(-i q_{g}\left(z_{g}+z_{s}\right)\right)$ cancel leaving

$$
\begin{equation*}
\int_{-\infty}^{\infty} d z^{\prime} \alpha_{2}\left(z^{\prime}\right) e^{2 i q_{g} z^{\prime}}=\frac{i k_{0}^{2}}{q_{g}} \int_{-\infty}^{\infty} d z^{\prime} \alpha_{1}\left(z^{\prime}\right) e^{2 i q_{g} z^{\prime}} \int_{-\infty}^{\infty} d z^{\prime \prime} H\left(z^{\prime}-z^{\prime \prime}\right) \alpha_{1}\left(z^{\prime \prime}\right) \tag{82}
\end{equation*}
$$

Integrating the right-hand side by parts

$$
\begin{aligned}
u & =\int_{-\infty}^{\infty} d z^{\prime \prime} \alpha_{1}\left(z^{\prime \prime}\right) H\left(z^{\prime}-z^{\prime \prime}\right) \alpha_{1}\left(z^{\prime}\right) \\
d v & =\frac{i k_{0}^{2}}{q_{g}} e^{2 i q_{g} z^{\prime}} d z^{\prime} \\
\frac{d u}{d z^{\prime}} & =\alpha_{1}^{2}\left(z^{\prime}\right)+\int_{-\infty}^{\infty} d z^{\prime \prime} \alpha_{1}\left(z^{\prime \prime}\right) H\left(z^{\prime}-z^{\prime \prime}\right) \frac{d \alpha_{1}\left(z^{\prime}\right)}{d z^{\prime}} \\
& =\alpha_{1}^{2}\left(z^{\prime}\right)+\int_{-\infty}^{z^{\prime}} d z^{\prime \prime} \alpha_{1}\left(z^{\prime \prime}\right) \frac{d \alpha_{1}\left(z^{\prime}\right)}{d z^{\prime}} \\
v & =\frac{k_{0}^{2}}{2 q_{g}^{2}} e^{2 i q_{g} z^{\prime}}
\end{aligned}
$$

Therefore,

$$
\begin{gather*}
\int_{-\infty}^{\infty} d z^{\prime} e^{2 i q_{g} z^{\prime}} \alpha_{2}\left(z^{\prime}\right)=\left[\frac{k_{0}^{2}}{2 q_{g}^{2}} e^{2 i q_{g} z^{\prime}} \int_{-\infty}^{\infty} d z^{\prime \prime} \alpha_{1}\left(z^{\prime \prime}\right) H\left(z^{\prime}-z^{\prime \prime}\right) \alpha_{1}\left(z^{\prime}\right)\right]_{z^{\prime}=-\infty}^{\infty} \\
-\int_{-\infty}^{\infty} d z^{\prime} e^{2 i q_{g} z^{\prime}} \frac{k_{0}^{2}}{2 q_{g}^{2}}\left(\alpha_{1}^{2}\left(z^{\prime}\right)+\int_{-\infty}^{z^{\prime}} d z^{\prime \prime} \alpha_{1}\left(z^{\prime \prime}\right) \frac{d \alpha_{1}\left(z^{\prime}\right)}{d z^{\prime}}\right) \tag{83}
\end{gather*}
$$

the boundary terms are zero (assuming $\alpha_{1}$, like $\alpha$, is confined to a finite region) and so

$$
\begin{equation*}
\tilde{\alpha}_{2}\left(-2 q_{g}\right)=-\int_{-\infty}^{\infty} d z^{\prime} e^{2 i q_{g} z^{\prime}} \frac{k_{0}^{2}}{2 q_{g}^{2}}\left(\alpha_{1}^{2}\left(z^{\prime}\right)+\int_{-\infty}^{z^{\prime}} d z^{\prime \prime} \alpha_{1}\left(z^{\prime \prime}\right) \frac{d \alpha_{1}\left(z^{\prime}\right)}{d z^{\prime}}\right) . \tag{84}
\end{equation*}
$$

Performing an inverse Fourier transform and holding the angle of incidence constant gives

$$
\begin{equation*}
\alpha_{2}\left(z, \theta_{0}\right)=-\frac{1}{2 \cos ^{2} \theta_{0}}\left(\alpha_{1}^{2}\left(z, \theta_{0}\right)+\int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime} \frac{\partial \alpha_{1}\left(z, \theta_{0}\right)}{\partial z}\right) \tag{85}
\end{equation*}
$$

## C Isolation of the leading order imaging portion from the third term, $\alpha_{3}$

For a detailed derivation and separation of the third term in the inverse series, the reader is referred to the appendices of Shaw et al. (2003). For the purposes of this paper, we include
only the steps taken to get to the generalized constant $\theta_{0}$ form. The integral equation to be solved for the third term in the series is

$$
\begin{align*}
\int_{-\infty}^{\infty} d \vec{x}^{\prime} G_{0}\left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right) k_{0}^{2} \alpha_{3}\left(z^{\prime}\right) & G_{0}\left(\vec{x}^{\prime} \mid \vec{x}_{s} ; \omega\right) \\
= & -\int_{-\infty}^{\infty} d \vec{x}^{\prime} G_{0}\left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) \\
& \times \int_{-\infty}^{\infty} d \vec{x}^{\prime \prime} G_{0}\left(\vec{x}^{\prime} \mid \vec{x}^{\prime \prime} ; \omega\right) k_{0}^{2} \alpha_{2}\left(z^{\prime \prime}\right) G_{0}\left(\vec{x}^{\prime \prime} \mid \vec{x}_{s} ; \omega\right) \\
- & \int_{-\infty}^{\infty} d \vec{x}^{\prime} G_{0}\left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right) k_{0}^{2} \alpha_{2}\left(z^{\prime}\right) \\
& \times \int_{-\infty}^{\infty} d \vec{x}^{\prime \prime} G_{0}\left(\vec{x}^{\prime} \mid \vec{x}^{\prime \prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(z^{\prime \prime}\right) G_{0}\left(\vec{x}^{\prime \prime} \mid \vec{x}_{s} ; \omega\right) \\
- & \int_{-\infty}^{\infty} d \vec{x}^{\prime} G_{0}\left(\vec{x}_{g} \mid \vec{x}^{\prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(z^{\prime}\right) \\
& \times \int_{-\infty}^{\infty} d \vec{x}^{\prime \prime} G_{0}\left(\vec{x}^{\prime} \mid \vec{x}^{\prime \prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(z^{\prime \prime}\right) \\
& \times \int_{-\infty}^{\infty} d \vec{x}^{\prime \prime \prime} G_{0}\left(\vec{x}^{\prime \prime} \mid \vec{x}^{\prime \prime \prime} ; \omega\right) k_{0}^{2} \alpha_{1}\left(z^{\prime \prime \prime}\right) G_{0}\left(\vec{x}^{\prime \prime \prime} \mid \vec{x}_{s} ; \omega\right) . \tag{86}
\end{align*}
$$

Fourier transform both sides of (86) over $x_{g}$ and $y_{g}$ and following the same steps as in deriving (74), the left-hand side of (86) becomes

$$
\begin{equation*}
L H S \rightarrow \frac{-k_{0}^{2}}{4 q_{g}^{2}} e^{-i k_{x_{g}} x_{s}} e^{-i k_{y_{g}} y_{s}} e^{-i q_{g}\left(z_{g}+z_{s}\right)} \int_{-\infty}^{\infty} \alpha_{3}\left(z^{\prime}\right) e^{2 i q_{g} z^{\prime}} d z^{\prime} \tag{87}
\end{equation*}
$$

Meanwhile, the right-hand side becomes

$$
\begin{align*}
R H S \rightarrow & -\frac{i k_{0}^{4}}{4 q_{g}^{3}} e^{-i k_{x_{g} x_{s}}} e^{-i k_{y_{g} y_{s}}} e^{-i q_{g}\left(z_{g}+z_{s}\right)} \int_{-\infty}^{\infty} d z^{\prime} \underline{\alpha_{1}}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d z^{\prime \prime} H\left(z^{\prime}-z^{\prime \prime}\right) \underline{\alpha_{2}}\left(z^{\prime \prime}\right) e^{2 i q_{g} z^{\prime}} \\
& -\frac{i k_{0}^{4}}{4 q_{g}^{3}} e^{-i k_{x_{g} x_{s}}} e^{-i k_{y_{g} y_{s}}} e^{-i q_{g}\left(z_{g}+z_{s}\right)} \int_{-\infty}^{\infty} d z^{\prime} \underline{\alpha_{2}}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d z^{\prime \prime} H\left(z^{\prime}-z^{\prime \prime}\right) \underline{\alpha_{1}}\left(z^{\prime \prime}\right) e^{2 i q_{g} z^{\prime}} \\
& -\frac{k_{0}^{4}}{(2 \pi)^{4}} e^{-i k_{x_{g} x_{s}}} e^{-i k_{y_{g}} y_{s}} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d k_{z^{\prime}} \frac{e^{i k_{z^{\prime}}\left(z_{g}-z^{\prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime}}^{2}} \\
& \times \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d z^{\prime \prime} \int_{-\infty}^{\infty} d k_{z^{\prime \prime}} \frac{e^{i k_{z^{\prime \prime}}\left(z^{\prime}-z^{\prime \prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z^{\prime \prime}}^{2}} \alpha_{1}\left(z^{\prime \prime}\right) \\
& \times \int_{-\infty}^{\infty} d z^{\prime \prime \prime} \int_{-\infty}^{\infty} d k_{z^{\prime \prime \prime}} \frac{e^{i k_{z^{\prime \prime \prime}}\left(z^{\prime \prime}-z^{\prime \prime \prime}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z_{g}}^{2}} \alpha_{1}\left(z^{\prime \prime \prime}\right) \int_{-\infty}^{\infty} d k_{z_{s}} \frac{e^{i k_{z_{s}}\left(z^{\prime \prime \prime}-z_{s}\right)}}{k_{0}^{2}-k_{x_{g}}^{2}-k_{y_{g}}^{2}-k_{z_{s}}^{2}} . \tag{88}
\end{align*}
$$

Simplifying gives

$$
\begin{align*}
\int_{-\infty}^{\infty} \alpha_{3}\left(z^{\prime}\right) e^{2 i q_{g} z^{\prime}} d z^{\prime}= & \frac{i k_{0}^{2}}{q_{g}} \int_{-\infty}^{\infty} d z^{\prime} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d z^{\prime \prime} H\left(z^{\prime}-z^{\prime \prime}\right) \alpha_{2}\left(z^{\prime \prime}\right) e^{2 i q_{g} z^{\prime}} \\
& \frac{i k_{0}^{2}}{q_{g}} \int_{-\infty}^{\infty} d z^{\prime} \alpha_{2}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d z^{\prime \prime} H\left(z^{\prime}-z^{\prime \prime}\right) \alpha_{1}\left(z^{\prime \prime}\right) e^{2 i q_{g} z^{\prime}} \\
& +\frac{k_{0}^{2}}{4 q_{g}^{2}} \int_{-\infty}^{\infty} d z^{\prime} e^{i q_{g} z^{\prime}} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} d z^{\prime \prime} e^{i q_{g}\left|z^{\prime}-z^{\prime \prime \prime}\right|} \alpha_{1}\left(z^{\prime \prime}\right) \\
& \int_{-\infty}^{\infty} d z^{\prime \prime \prime} e^{i q_{g} \mid z^{\prime \prime}-z^{\prime \prime \prime}} \mid \alpha_{1}\left(z^{\prime \prime \prime}\right) e^{i q_{g} z^{\prime \prime \prime}} . \tag{89}
\end{align*}
$$

Then, integrating by parts and inverse Fourier transforming holding $\theta_{0}$ constant gives

$$
\begin{align*}
\alpha_{3}\left(z, \theta_{0}\right)= & \frac{-k_{0}^{2}}{q_{g}^{2}} \alpha_{1}\left(z, \theta_{0}\right) \alpha_{2}\left(z, \theta_{0}\right)-\frac{-k_{0}^{2}}{2 q_{g}^{2}} \alpha_{1}\left(z, \theta_{0}\right) \int_{-\infty}^{z} \alpha_{2}\left(z^{\prime}, \theta_{0}\right) d z^{\prime} \frac{\partial \alpha_{1}\left(z, \theta_{0}\right)}{\partial z} \\
& -\frac{k_{0}^{2}}{2 q_{g}^{2}} \alpha_{2}\left(z, \theta_{0}\right) \int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime} \frac{\partial \alpha_{2}\left(z, \theta_{0}\right)}{\partial z} \\
& -\frac{k_{0}^{2}}{4 q_{g}^{2}} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d z^{\prime \prime} \int_{-\infty}^{\infty} d z^{\prime \prime \prime} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) \alpha_{1}\left(z^{\prime \prime}, \theta_{0}\right) \alpha_{1}\left(z^{\prime \prime}, \theta_{0}\right) \\
& \times\left(H\left(z^{\prime}-z^{\prime \prime}\right) H\left(z^{\prime \prime}-z^{\prime \prime \prime}\right) e^{2 i i_{g} z^{\prime}} e^{-i q_{g} z^{\prime \prime \prime}}\right. \\
& +H\left(z^{\prime}-z^{\prime \prime}\right) H\left(z^{\prime \prime \prime}-z^{\prime \prime}\right) e^{i q_{g} z^{\prime}} e^{-2 i q_{g} z^{\prime \prime}} e^{i q_{g} z^{\prime \prime \prime}} \\
& +H\left(z^{\prime \prime}-z^{\prime}\right) H\left(z^{\prime \prime}-z^{\prime \prime \prime}\right) e^{-i q_{g} z^{\prime}} e^{2 i q_{g} z^{\prime \prime}} e^{-i q_{g} z^{\prime \prime \prime}} \\
& \left.+H\left(z^{\prime \prime}-z^{\prime}\right) H\left(z^{\prime \prime \prime}-z^{\prime \prime}\right) e^{-i q_{g} z^{\prime}} e^{i q_{g} z^{\prime \prime \prime}}\right) . \tag{90}
\end{align*}
$$

By comparison with the 1-D normal incidence derivation (Shaw et al., 2003; Innanen, 2003),

$$
\begin{align*}
\alpha_{3}\left(z, \theta_{0}\right)= & \frac{1}{\cos ^{4} \theta_{0}}\left(\frac{3}{16} \alpha_{1}^{3}\left(z, \theta_{0}\right)+\frac{1}{8}\left(\int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}\right)^{2}\left[\frac{\partial^{2}}{\partial z^{2}} \alpha_{1}\left(z, \theta_{0}\right)\right]\right. \\
& +\frac{5}{8} \alpha_{1}\left(z, \theta_{0}\right) \int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}\left[\frac{\partial}{\partial z} \alpha_{1}\left(z, \theta_{0}\right)\right] \\
& +\frac{1}{8}\left[\frac{\partial}{\partial z} \alpha_{1}\left(z, \theta_{0}\right)\right] \int_{-\infty}^{z}\left(\int_{-\infty}^{z^{\prime}} \alpha_{1}\left(z^{\prime \prime}, \theta_{0}\right) d z^{\prime \prime}\left[\frac{\partial}{\partial z^{\prime}} \alpha_{1}\left(z^{\prime}, \theta_{0}\right)\right]\right) d z^{\prime} \\
& \left.-\frac{1}{16} \int_{-\infty}^{z} \int_{-\infty}^{z}\left[\frac{\partial}{\partial z^{\prime}} \alpha_{1}\left(z^{\prime}, \theta_{0}\right)\right]\left[\frac{\partial}{\partial z^{\prime \prime}} \alpha_{1}\left(z^{\prime \prime}, \theta_{0}\right)\right] \alpha_{1}\left(z^{\prime \prime}+z^{\prime}-z, \theta_{0}\right) d z^{\prime \prime} d z^{\prime}\right) \tag{91}
\end{align*}
$$

and the amplitude-only and leading order imaging contributions are

$$
\begin{equation*}
\alpha_{3}\left(z, \theta_{0}\right)=\frac{3}{16} \frac{1}{\cos ^{4} \theta_{0}} \alpha_{1}^{3}\left(z, \theta_{0}\right)+\frac{1}{8} \frac{1}{\cos ^{4} \theta_{0}}\left(\int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}\right)^{2} \frac{\partial^{2} \alpha_{1}\left(z, \theta_{0}\right)}{\partial z^{2}}+\cdots \tag{92}
\end{equation*}
$$

respectively.

# A leading order imaging series for prestack data acquired over a laterally invariant acoustic medium. Part II: Analysis for data missing low frequencies 

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#### Abstract

The leading order imaging series images reflectors closer to their correct spatial location than a conventional depth imaging algorithm using the same reference velocity and without solving for the actual velocity model. The algorithm is a subseries of the inverse scattering series, a direct multidimensional inversion procedure, and is nonlinear in the measured wavefield. In this paper, we continue the progression of testing the effectiveness of the prestack leading order imaging series (Part I) for limited input data, specifically when the input data are missing low temporal frequency information, which is invariably the case in the seismic experiment.

It is demonstrated that, while it benefits from low frequency information, the leading order imaging series retains effectiveness even when zero and low frequency information are absent. Furthermore, its effectiveness is enhanced with offset data where the conjugate to depth $k_{z}=2\left(\omega / c_{0}\right) \cos \theta$ replaces $k_{z}=2\left(\omega / c_{0}\right)$ at normal incidence, allowing for a lower minimum $k_{z}$ than is possible at normal incidence for the same temporal frequency bandwidth.

The relationship to low frequency information is analyzed by examining the imaging series' integral with respect to depth of the first term. It is shown that the fidelity of the construction of the first term in the low $k_{z}$ band is important. However, the integration limits of the integral in the algorithm are not $-\infty$ to $+\infty$ and so the imaging series does not call for the zero frequency component of the data. We demonstrate using numerical reflectivity data examples, that even when missing zero and low temporal frequency information, the leading order imaging series improves the predicted depth of reflectors over conventional imaging with the reference velocity. While the imaging series is effective even when low frequencies are absent, greater effectiveness can be achieved when lower frequency information is present, which merits the study of existing and new spectral extrapolation techniques (Innanen et al., 2004), and is in perfect alignment with the current trend to acquisition systems that record lower frequency data.

These results are encouraging and justify the current progression of this and other inverse subseries algorithms to a multidimensional earth (Liu et al., 2004) and elastic wave equations (Zhang and Weglein, 2004).


## 1 Introduction

The development of inverse scattering series algorithms for seismic data processing begins with the isolation of a subseries that accomplishes a specific task, for example, free surface multiple removal, internal multiple attenuation or depth imaging (Weglein et al., 2003). Once a candidate subseries has been identified for the simplest earth and acquisition models, it is systematically analyzed under progressively more realistic conditions with the objective of developing it into an algorithm ready for field data.

The inverse scattering series, a multidimensional direct inversion procedure, is non-linear in the measured data. Other non-linear inverse methods (for example, Tarantola, 1987) have been found to fail in practice because field data are always band-limited, and missing low frequencies precludes a successful updating of the reference model towards the actual model. One important distinction between the inverse scattering series and iterative linear inversion is that, with the inverse series, the reference model is never updated. Every term in the inverse series is computed through an inversion of a dataset using only the original reference medium properties. Furthermore, the inverse subseries and task separation approach (Weglein et al., 2003), which inverts seismic data one step at a time, is in contrast to linear iterative inversion, which uses the total wavefield to directly invert for earth properties at once.

Despite significant differences between the inverse scattering subseries procedure and other non-linear inversion methods, it might be expected that at least one of the subseries algorithms would benefit from low frequency information. After all, they achieve seismic processing objectives without subsurface information by engaging the data more fully than algorithms that expect accurate a priori details about the medium. If the algorithms expect more from the data, then it is reasonable to think that at least one of them might benefit from a broad frequency spectrum. According to the convolutional model, a single frequency component from the source will experience the medium in a manner described by a linear differential operator before it is recorded at the receiver. The convolutional model applies to even the most complex absorptive elastic wave equations currently used to describe wave propagation. However, direct inversion of even the simplest single-parameter acoustic wave equation using the inverse scattering series requires communication between different frequencies.

The objective of this paper is to analyze the role of low frequency information in the leading order imaging series and, specifically, to answer the question of whether it retains any benefit when low frequency information is absent from the data. In preparation for the analysis, Shaw and Weglein (2004) derived this imaging series for prestack input data (Part I). This form of the algorithm expects input data that more closely represent the actual seismic experiment than a normal incidence algorithm because the source-receiver offset leads to a lower vertical wavenumber, $k_{z}$. The issue of missing low frequency in non-linear inversion is in fact one of missing low $k_{z}$, rather than missing low temporal frequency, $\omega$.

3-D wave propagation in a constant density acoustic medium that varies only in 1-D (see


Figure 1: A multi-layer 1-D constant density acoustic model.
Fig. 1) is characterized by

$$
\begin{equation*}
\left(\nabla^{2}+\frac{\omega^{2}}{c_{0}^{2}}(1-\alpha(z))\right) P\left(\vec{x} \mid \vec{x}_{s} ; \omega\right)=-A(\omega) \delta\left(\vec{x}-\vec{x}_{s}\right) \tag{1}
\end{equation*}
$$

where $P$ is the pressure field, $A$ is the source wavelet, $\omega$ is the angular temporal frequency, $c_{0}$ is the reference velocity and $\alpha$ is the velocity perturbation. In (1), the actual medium velocity, $c$, has been expressed in terms of a constant reference velocity, $c_{0}$, and the perturbation, $\alpha$, where

$$
\begin{equation*}
\frac{1}{c^{2}(z)}=\frac{1}{c_{0}^{2}}(1-\alpha(z)) \tag{2}
\end{equation*}
$$

The solution for $\alpha$ can be written as an infinite series

$$
\begin{equation*}
\alpha=\alpha_{1}+\alpha_{2}+\alpha_{3}+\cdots \tag{3}
\end{equation*}
$$

where $\alpha_{1}$, the first term in the inverse series for $\alpha$, is linearly related to the scattered field, $P_{s}=P-P_{0} . P_{0}$ is the pressure wavefield due to the same source, $A(\omega)$, in the reference medium, chosen to be a wholespace with velocity $c_{0}$. The second term, $\alpha_{2}$, is quadratic in $P_{s}$, the third term, $\alpha_{3}$, is cubic and so on.
The imaging series is responsible for positioning reflectors at their correct spatial location (Weglein et al., 2000, 2002). The objective of the imaging series, $\alpha^{\mathrm{IM}}$, a subseries of the
inverse series (3), is to solve directly for the location at which $\alpha$ changes. In other words, to solve for $\alpha^{\mathrm{IM}}$ is to solve the problem of imaging in a medium whose velocity is not known before or after the imaging procedure. The leading order imaging series, $\alpha^{\text {LOIM }}$, is the contribution to the imaging series that is leading order in the amplitudes of the scattered field (Shaw et al., 2003).

For the medium in which equation (1) describes wave propagation, and for a constant reference medium angle of incidence $\theta_{0}$, the leading order imaging series can be written as an infinite sum of operations on the first term (Shaw and Weglein, 2004)

$$
\alpha^{\mathrm{LOIM}}\left(z, \theta_{0}\right)=\sum_{n=0}^{\infty} \frac{(-1 / 2)^{n}}{\frac{\left(-\cos ^{2 n} \theta_{0}\right.}{n+c^{\prime}}} \times \underbrace{\left(\int_{0}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}\right)^{n}}_{\text {Scalar }} \times \underbrace{\frac{d^{n} \alpha_{1}\left(z, \theta_{0}\right)}{d z^{n}}}_{\begin{array}{c}
\text { Emphasizes low }  \tag{4}\\
\text { frequency }
\end{array}}
$$

where $\alpha_{1}$ is computed through a scaled slant stack of the data, and in cylindrical coordinates is

$$
\begin{equation*}
\alpha_{1}\left(z, \theta_{0}\right)=-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{0}^{2 \pi} \int_{0}^{\infty} D\left(r ; \tau_{0}-p_{0} r \cos \phi\right) r d r d \phi \tag{5}
\end{equation*}
$$

The data are the scattered field with the source wavelet deconvolved $\left(D=P_{s} / A\right), r$ is the source-receiver offset in the horizontal plane, $\phi$ is the azimuth, $p_{0}$ is the reference horizontal slowness, and $\zeta_{0}$ is the reference vertical slowness.

In (4), we have labelled the parts of the algorithm according to their effect on the frequency spectrum of $\alpha^{\text {LOIM }}$. The imaging series is a Taylor series for a box (i.e., the difference of two Heaviside functions) expanded about the depth of each mislocated interface. The coefficients of each term in the Taylor series are the product of the scalar with the integral of $\alpha_{1}$. In the $k_{z}$ domain, (4) is a power series for an exponential function. Recognizing this fact, then for a 1-D acoustic medium the leading order imaging series has a closed form solution:

$$
\begin{equation*}
\alpha^{\mathrm{LOIM}}\left(z, \theta_{0}\right)=\alpha_{1}\left(z-1 /\left(2 \cos ^{2} \theta_{0}\right) \int_{0}^{z} \alpha_{1}\left(z^{\prime}, \theta_{0}\right) d z^{\prime}, \theta_{0}\right) \tag{6}
\end{equation*}
$$

This form will be used for the analysis in this paper. Studying the closed form focuses attention on the performance of the algorithm itself and not the numerical implementation of its series form. From (6), it is clear that the shift of each interface is proportional to the integral of $\alpha_{1}$ above the reflector being imaged. This integral captures amplitude and residual moveout information in the overburden of each reflector and emphasizes $\tilde{\alpha}_{1}$ at low $k_{z}$ values (because it is a division by $k_{z}$ ). The fidelity of $\alpha_{1}$ and its integral will have a direct impact on the performance of the imaging series. Therefore, it is instructive to review the construction of $\tilde{\alpha}_{1}\left(k_{z}\right)$ from the data.


Figure 2: The dispersion relationship between the vertical and horizontal wavenumbers and temporal frequency.

## 2 Construction of the first term, $\alpha_{1}$

In the wavenumber domain, the first term in the inverse series for the single parameter acoustic problem and a constant velocity reference medium is

$$
\begin{equation*}
\tilde{\alpha}_{1}\left(k_{z}\right)=2 \pi \frac{-k_{z}^{2}}{k_{0}^{2}} e^{-i k_{z}\left(z_{g}+z_{s}\right) / 2} \int_{0}^{\infty} \tilde{D}(r ; \omega) J_{0}\left(k_{r} r\right) r d r \tag{7}
\end{equation*}
$$

where $k_{0}=\omega / c_{0}$ and the vertical and horizontal wavenumbers, $k_{z}$ and $k_{r}$, respectively, are related by (see Fig. 2)

$$
\begin{equation*}
k_{z}=-2 \frac{\omega}{c_{0}} \sqrt{1-\frac{k_{r}^{2} c_{0}^{2}}{\omega^{2}}}=-2 \omega \zeta_{0} . \tag{8}
\end{equation*}
$$

$J_{0}\left(k_{r} r\right)$ is a zero order Bessel function of the first kind that arises due to the azimuthal symmetry and is

$$
\begin{equation*}
J_{0}\left(k_{r} r\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{i k_{r} r \cos \phi^{\prime}} d \phi^{\prime} . \tag{9}
\end{equation*}
$$

The recorded data have spatial, temporal and wavenumber apertures. To keep the current discussion to the effect of missing low temporal frequency, it is assumed for the moment that the source-receiver offset aperture extends from $r=0 \rightarrow \infty$.

By virtue of the source-receiver offset, the problem of determining $\tilde{\alpha}_{1}$ is over-determined since there are more free variables on the right-hand side of (7) than on the left. Hence, there is more than one way to compute $\tilde{\alpha}_{1}\left(k_{z}\right)$. For example, to compute $\tilde{\alpha}_{1}\left(k_{z}=0\right)$, we


Figure 3: The constraints imposed by bandlimited $\omega$ for two different angles of incidence, $\theta_{0}^{1}$ and $\theta_{0}^{2}$. Both $\left(k_{z}\right)_{\min }$ and $\left(k_{z}\right)_{\max }$ will be smaller for larger angles.
can either set $\omega=0$ or $k_{r}=\omega / c_{0}$. Since $k_{z}=-2 k_{0} \cos \theta_{0}$, then $k_{z} \rightarrow 0$ when $\theta_{0} \rightarrow 90^{\circ}$. Large angles of incidence can construct $\tilde{\alpha}_{1}$ at low $k_{z}$ values. Inverse Fourier transforming both sides of (7) gives

$$
\begin{align*}
\alpha_{1}(z) & =\frac{1}{2 \pi} \int_{-\infty}^{\infty} \tilde{\alpha}_{1}\left(k_{z}\right) e^{i k_{z} z} d k_{z} \\
& =-\int_{-\infty}^{\infty} \frac{k_{z}^{2}}{k_{0}^{2}} e^{i k_{z}\left(z-\left(z_{g}+z_{s}\right) / 2\right)} \int_{0}^{\infty} \tilde{D}(r ; \omega) J_{0}\left(k_{r} r\right) r d r d k_{z} . \tag{10}
\end{align*}
$$

Given the degree of freedom in the data, there is a choice regarding how to evaluate (10). Fixing the angle of incidence, $\theta_{0}$, and integrating over angular frequency, $\omega$, leads to different estimates of $\alpha_{1}$ for each $\theta_{0}$, denoted by $\alpha_{1}\left(z, \theta_{0}\right)$. This parameterization is illustrated in Fig. 3. Fixing $\theta_{0}$ is the same as fixing horizontal and vertical slownesses, $p_{0}$ and $\zeta_{0}$, respectively,

## (a)

(b)


Figure 4: The constraints imposed by bandlimited $\omega$ on the bandwidth of $k_{z}$ when integrating over angle. This constant-frequency parametrization is an alternative to the one illustrated in Fig. 3 and is still under investigation.
where

$$
p_{0}=\frac{\sin \theta_{0}}{c_{0}} \text { and } \zeta_{0}=\frac{\cos \theta_{0}}{c_{0}} .
$$

An alternative approach to handling the degree of freedom in (10) is to keep $\omega$ fixed and integrate over angle or vertical slowness, $\zeta_{0}$. This parameterization (illustrated in Fig. 4) will result in different estimates of $\alpha_{1}(z)$ for constant $\omega$ values and is the subject of ongoing research.

In this paper, we choose to hold $\theta_{0}$ fixed which allows $\left(k_{z} / k_{0}\right)^{2}=4 \cos ^{2} \theta_{0}$ to come out of the outer integral in (10). Then, changing variables from $k_{z}$ to $\omega$ (recalling $k_{z}=-2 \omega \zeta_{0}$ ) gives

$$
\begin{equation*}
\alpha_{1}\left(z, \theta_{0}\right)=-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{-\infty}^{\infty} e^{-i \omega \zeta_{0}\left(2 z-\left(z_{g}+z_{s}\right)\right)} \int_{0}^{\infty} \tilde{D}(r ; \omega) J_{0}\left(\omega p_{0} r\right) r d r d \omega \tag{11}
\end{equation*}
$$

Defining $\tau_{0}=\zeta_{0}\left(2 z-\left(z_{g}+z_{s}\right)\right)$ and performing the inverse temporal Fourier transform of the data $\tilde{D}(r ; \omega)$, (11) becomes

$$
\begin{align*}
\alpha_{1}\left(z, \theta_{0}\right) & =-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{0}^{2 \pi} \int_{0}^{\infty} D\left(r ; \tau_{0}-p_{0} r \cos \phi\right) r d r d \phi  \tag{12}\\
& =-8 \zeta_{0} \cos ^{2} \theta_{0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D\left(x, y ; \tau_{0}-x p_{0}\right) d x d y \tag{13}
\end{align*}
$$

which is recognizable as a scaled slant stack of the data (Treitel et al., 1982). As has been mentioned, it is assumed here that the data have infinite spatial aperture, so that the integrals in (12) and (13) can be evaluated accurately.

## 3 The effect of bandlimited data on the leading order imaging series at normal incidence

The data for a model that has two horizontal interfaces at depths $z_{a}$ and $z_{b}$ consist of two primary reflections. For a single frequency component,

$$
\begin{equation*}
\tilde{D}(r ; \omega)=-\int_{0}^{\infty} \frac{R_{01}+R_{12}^{\prime} e^{2 i \omega \zeta_{1}\left(z_{b}-z_{a}\right)}}{i \omega \zeta_{0}} e^{i \omega \zeta_{0}\left(2 z_{a}-z_{s}-z_{g}\right)} J_{0}\left(k_{r} r\right) k_{r} d k_{r} \tag{14}
\end{equation*}
$$

where the reflection and transmission coefficients are functions of angle and are given by

$$
\begin{equation*}
R_{01}=\frac{\zeta_{0}-\zeta_{1}}{\zeta_{0}+\zeta_{1}}, R_{12}^{\prime}=T_{01} R_{12} T_{10}, T_{01}=\frac{-2 \zeta_{1}}{\zeta_{0}+\zeta_{1}}, R_{12}=\frac{\zeta_{1}-\zeta_{2}}{\zeta_{1}+\zeta_{2}} \text { and } T_{10}=\frac{2 \zeta_{0}}{\zeta_{0}+\zeta_{1}} \tag{15}
\end{equation*}
$$

The vertical slownesses are functions of the incident angles in each layer according to

$$
\begin{equation*}
\zeta_{i}=\frac{\cos \theta_{i}}{c_{i}}, i=0,1,2, \ldots \tag{16}
\end{equation*}
$$

Substituting the data (14) into the linear inverse equation (11), the first term in the series for $\alpha\left(z, \theta_{0}\right)$ is a sum over all frequencies

$$
\begin{equation*}
\alpha_{1}\left(z, \theta_{0}\right)=8 \cos ^{2} \theta_{0} \int_{-\infty}^{\infty}\left(\frac{R_{01}+R_{12}^{\prime} e^{2 i \omega \zeta_{1}\left(z_{b}-z_{a}\right)}}{i \omega} e^{2 i \omega \zeta_{0}\left(z_{a}-z\right)}\right) d \omega . \tag{17}
\end{equation*}
$$

Given all temporal frequencies, then (17) produces two Heaviside functions, one for each reflector:

$$
\begin{equation*}
\alpha_{1}\left(z, \theta_{0}\right)=4 \cos ^{2} \theta_{0}\left[R_{01}\left(\theta_{0}\right) H\left(z-z_{a}\right)+R_{12}^{\prime}\left(\theta_{0}\right) H\left(z-z_{b^{\prime}}\right)\right] \tag{18}
\end{equation*}
$$



Figure 5: In practice, $\omega$ will be bandlimited between $\omega_{\min }$ and $\omega_{\max }$.
where the shallower reflector is correctly located at $z_{a}$ (since the velocity down to $z_{a}$ was correct) but the deeper reflector is mislocated at depth

$$
\begin{equation*}
z_{b^{\prime}}=z_{a}+\left(z_{b}-z_{a}\right) \frac{\zeta_{1}}{\zeta_{0}} \tag{19}
\end{equation*}
$$

In (18), it is emphasized that the amplitudes of each reflection event are functions of angle.
We proceed by first considering the special case of normal incidence, i.e., $\theta_{0}=p_{0}=0$. The effect on $\alpha_{1}(z)$ when the $\omega$ spectrum is bandlimited (Fig. 5) is to produce bandlimited Heaviside functions. This is illustrated in Fig. 6 for a particular model ( $c_{0}=1500 \mathrm{~m} / \mathrm{sec}$, $c_{1}=1650 \mathrm{~m} / \mathrm{sec}, c_{2}=1510 \mathrm{~m} / \mathrm{sec}, z_{a}=1000 \mathrm{~m}, z_{b^{\prime}}=1075 \mathrm{~m}$ ) and a range of different minimum frequencies $\left(f_{\min }=\omega_{\min } /(2 \pi)=0,1,2,3 \mathrm{~Hz}\right)$. Plotted beside each $\alpha_{1}\left(z, \theta_{0}=0\right)$ is its integral, which is an indication of how the leading order imaging series shift will be affected by missing low frequency information. As low frequency information is erased from the input data, the integral of $\alpha_{1}$ deviates from its analytically computed values for infinite bandwidth. This error will impact the shift computed by the leading order imaging series as shown in Fig. 7. Even though the data contain no information below 1 Hz , the second interface still shifts towards the correct depth $z_{b}$ because the integral of $\alpha_{1}$ is a fair approximation to the exact value computed with infinite bandwidth. As a result of the error in the integral for $z \leq z_{a}$, the reflector that was correctly imaged at $z_{a}$ by $\alpha_{1}$, is shifted slightly to a shallower depth.

The results in Fig. 7 can be significantly improved by recognizing that the area under the $\alpha_{1}$ curve for $z<z_{a}$ is largely responsible for the error in the integral of $\alpha_{1}$ at $z>z_{a}$ (see Fig. 6). It is known a priori that the perturbation $\alpha\left(z<z_{a}\right)=0$ (and, by extension, $\alpha_{1}\left(z<z_{a}\right)=0$ ) because the reference medium agrees with the actual medium at the measurement surface. It is not physically possible for the perturbation to be non-zero before the onset of the recorded reflectivity. This effect of missing low frequency can be straightforwardly rectified: rather than integrate from the measurement surface through an $\alpha_{1}$ that is obviously in error, we choose to begin computing the integral at some small distance $\epsilon$ above the first reflector whose location is well-defined. We can further improve the integral of $\alpha_{1}$, and impose a known condition, by fixing the value of the perturbation at $\alpha_{1}\left(z_{a}-\epsilon\right)$ to be zero and shifting all values of $\alpha_{1}$ for $z>z_{a}-\epsilon$ by the value at $z_{a}-\epsilon$. Figure 8 illustrates the significant improvement in $\alpha_{1}$ and its integral when making this simple correction. Implementing this causality-like condition in the leading order imaging series gives

$$
\begin{equation*}
\hat{\alpha}^{\mathrm{LOIM}}\left(z, \theta_{0}\right)=\alpha_{1}\left(z-1 /\left(2 \cos ^{2} \theta_{0}\right) \int_{z_{a}-\epsilon}^{z}\left(\alpha_{1}\left(z^{\prime}, \theta_{0}\right)-\alpha_{1}\left(z_{a}-\epsilon, \theta_{0}\right)\right) d z^{\prime}, \theta_{0}\right) . \tag{20}
\end{equation*}
$$



Figure 6: The effect of missing low frequencies on $\alpha_{1}(z)$ (left) and its integral (right). The thin black lines are analytically computed values of $\alpha_{1}$ and its integral for infinite bandwidth. The thick blue lines are the numerically computed values for bandlimited input data.


Figure 7: The effect of missing low frequencies on $\alpha_{1}(z)$ (in red) and $\alpha^{\operatorname{LOIM}}(z)$ (in green). Despite not having any information below 1 Hz , the second interface still shifts towards the correct depth $z_{b}$. There is a small error at the first interface which can be eradicated by changing the integration limits (Fig. 9).

In all subsequent examples, we implement the leading order imaging series by imposing the condition that the perturbation must be zero before the onset of the scattered field (20). Then, for the same example as in Fig. 7, Fig. 9 shows the improvement in the result of the leading order imaging series when provided with this new integral of $\alpha_{1}$.

Figures 10-12 show results of the leading order imaging series for a five-layer model ( $c_{0}=1500$ $\mathrm{m} / \mathrm{sec}, c_{1}=1600 \mathrm{~m} / \mathrm{sec}, c_{2}=1550 \mathrm{~m} / \mathrm{sec}, c_{3}=1625 \mathrm{~m} / \mathrm{sec}, c_{4}=1510 \mathrm{~m} / \mathrm{sec}, z_{a}=1000 \mathrm{~m}, z_{b}=1075$ $\mathrm{m}, z_{c}=1125 \mathrm{~m}, z_{d}=1200 \mathrm{~m}$ ) and a range of missing low frequencies up to 8 Hz . Figure 11 shows the same information as Fig. 10 except the derivatives with respect to $z$ are displayed in order to more clearly identify the location of the reflectors. As demonstrated in Fig. 12, even after removing all information below 8 Hz (in this example), the leading order imaging series shows an improvement in the location of all reflectors.

Reducing the contrast between the actual and reference media is akin to having a reference velocity model that is a better estimate of the actual velocity. The effect will be apparent in the first term, $\alpha_{1}$, because reducing the difference in the velocities means that the reflectors will be imaged closer to their true depths even before the non-linear terms are computed.


Figure 8: The effect of missing low frequencies on $\alpha_{1}(z)$ and its integral when changing the integration limits from $\int_{0}^{z}$ to $\int_{z_{a}-\epsilon}^{z}$. Compared with Fig. 6, the results are significantly better when constraining the perturbation.


Figure 9: By truncating the integral from $\int_{0}^{z}$ to $\int_{z_{a}-\epsilon}^{z}$, the major effect of missing low frequencies on $\alpha^{L O I M}(z)$ (in green) is greatly mitigated. In comparison with Fig. 7, the mislocated reflector has been accurately located by the imaging series.

This is illustrated Fig. 13 where the velocity contrasts of the model in the top panel (labelled "Highest contrast") have been increased by $50 \%$ over those in the middle panel (labelled "Middle contrast") which are $50 \%$ greater than those in the bottom panel (labelled "Lowest contrast"). In each case, the lower frequency limit is 4 Hz and the proportional correction to the depths of the mislocated reflectors is approximately the same. Therefore, an increase in contrast would seem to not be more sensitive to missing low frequencies. The obvious benefit of lower contrasts, however, is that the closer the reference velocity is to the actual velocity, the better the leading order approximation is in the imaging series and the faster the series form will converge (Shaw et al., 2003). This observation regarding contrast and low frequency content is relevant to the next step in our analysis, which is to consider higher angles of incidence, where the effective contrast is greater.


Figure 10: A five-layer model comparing $\alpha_{1}$ (in red) with $\alpha^{\text {LOIM }}$ (in green) for $\theta_{0}=0$ and different low frequency limits. The mislocated reflectors shift towards their true depths.

## 4 The effect of bandlimited data at higher angles of incidence

As illustrated in Fig. 3, the degree of freedom afforded by the seismic experiment's sourcereceiver offset leads to a lower minimum vertical wavenumber, $\left(k_{z}\right)_{\min }$, in the construction


Figure 11: A five-layer model comparing the derivatives with respect to $z$ of $\alpha_{1}$ (in red) and $\alpha^{\text {LOIM }}$ (in green) for $\theta_{0}=0$ and different low frequency limits. The mislocated reflectors shift towards their true depths.
of $\tilde{\alpha}_{1}$ for higher angles of incidence through the relationship

$$
\begin{equation*}
\left(k_{z}\right)_{\min }=\frac{(\omega)_{\min }}{c_{0}} \cos \left[\left(\theta_{0}\right)_{\max }\right] . \tag{21}
\end{equation*}
$$



Figure 12: A five-layer model comparing the derivatives with respect to $z$ of $\alpha_{1}$ (in red) and $\alpha^{\text {LOIM }}$ (in green) for $\theta_{0}=0$ and different low frequency limits (missing up to 8 Hz ). Although there is a gradual degradation of the results when more low frequency is missing, in all cases the mislocated reflectors shift towards their true depths providing an improvement over linear imaging with the reference velocity.

There are two factors that affect the amplitude of $\alpha_{1}\left(z, \theta_{0}\right)$ as a function of angle. The first is data-driven: the magnitude of the data's amplitudes (for these acoustic models) will tend to increase with angle. The second is algorithm-driven: the computation of $\alpha_{1}$ involves a multiplication by $\cos ^{2} \theta_{0}$. The net result, illustrated in Fig. 14 for the first interface (where there is no transmission loss in the recorded data), is that the amplitude of $\alpha_{1}\left(z, \theta_{0}\right)$ will tend to vary only gradually with angle. However, the leading order imaging series undoes the first term's algorithmic angle dependent scalar, $\cos ^{2} \theta_{0}$, because higher angles of incidence will tend to have more residual moveout that needs to be corrected (Fig. 15). It is the larger magnitudes of the amplitudes with higher angles (Fig. 14) that set about correcting the greater error in depth (Fig. 15) through the imaging series algorithm (Shaw and Weglein, 2004).

The relevant question in the current analysis is how does the performance of the imaging series vary with angle when low temporal frequencies are missing? The results displayed in Fig. 13 would suggest that the higher "effective contrast" experienced at higher angles would be neither more nor less sensitive to missing low frequencies. However, in the leading order approximation to the imaging series that is currently being studied, higher contrasts will directly impact the accuracy of the predicted depths (Shaw et al., 2003). Specifically, the smaller the difference between the actual and reference media velocities, the more accurately the leading order imaging series will predict the precise locations of the reflectors.
Figure 16 demonstrates how higher angles of incidence fill in the low end of the $k_{z}$ spectrum for $4-62.5 \mathrm{~Hz}$ bandlimited input data. At $\theta_{0}=0^{\circ}$, the $k_{z}$ spectrum derives no additional benefit at the low end in the sense that the $\left(k_{z}\right)_{\min }$ is equal to $(\omega)_{\min } / c_{0}$. However, at $\theta_{0}=45^{\circ},\left(k_{z}\right)_{\min }$ is reduced by the factor $\cos \theta_{0}$ and is equivalent to 2.8 Hz (as opposed to 4 Hz ) at normal incidence. At higher angles, there is more residual moveout to be corrected for the nonlinear imaging terms.

Figure 17 shows results of the imaging series, for a fixed angle of incidence $\left(\theta_{0}=45^{\circ}\right)$, and a range of minimum temporal frequencies. It is encouraging to see that the difference in the predicted depths for $f_{\min }=0.125 \mathrm{~Hz}$ and $f_{\min }=2 \mathrm{~Hz}$ are extremely close and there are improvements over the first term for all cases where information below at least 6 Hz is missing.

In Fig. 18, the effect of missing low frequency on the prestack leading order imaging series is demonstrated on angle gathers over a range of precritical angles. While a gradual deterioration in effectiveness is evident as more low frequency information is removed, in all cases (at least up to $f_{\min }=4 \mathrm{~Hz}$ ) for this example, the results of the imaging series are an improvement over conventional imaging with the reference velocity (i.e., the depths are more accurate than those predicted by the first term in the series).
It is interesting to consider whether the low $k_{z}$ information at high angles can be transplanted to low angles of incidence to improve the performance of the leading order imaging series in the latter range. Figure 19 shows the effect of transplanting the low end of the $k_{z}$ amplitude spectrum from $\alpha_{1}\left(z, \theta_{0}=50^{\circ}\right)$ to the normal incidence trace, $\alpha_{1}\left(z, \theta_{0}=0^{\circ}\right)$ over the range that it is missing and then computing the leading order imaging series. A small improvement
in the location of the deepest reflector is noticeable. This procedure resembles the alternative parameterization discussed earlier in which we can choose to keep omega constant and sum of angles to compute each term in the series, thereby collecting high angle information for each estimate of $\alpha^{\text {LOIM }}$. This parameterization and procedure are currently under investigation.

## 5 Discussion

Analysis of the leading order imaging series algorithm for a 1-D acoustic earth that varies only in the $z$-direction, would suggest that low frequency information is important to its effectiveness because of its integral with respect to $z$. However, the limits of the integral are the reason why the algorithm can tolerate missing low frequency while still providing benefit. Thankfully, the algorithm does not call for

$$
\begin{equation*}
\int_{-\infty}^{\infty} \alpha_{1}\left(z^{\prime}\right) d z^{\prime} \tag{22}
\end{equation*}
$$

which is exactly the zero frequency component of the data. We have investigated procedures for improving the actual integral

$$
\begin{equation*}
\int_{0}^{z} \alpha_{1}\left(z^{\prime}\right) d z^{\prime} \tag{23}
\end{equation*}
$$

which collects the amplitude and moveout information in the overburden and acts to shift the mislocated reflectors towards their true locations in depth. One effective procedure that improves the accuracy of this integral derives from recognizing that the perturbation is before the onset of a reflector (say, at depth $z_{a}$ ). Redefining the integral to be

$$
\begin{equation*}
\int_{z_{a}-\epsilon}^{z} \alpha_{1}\left(z^{\prime}\right) d z^{\prime} \tag{24}
\end{equation*}
$$

significantly improves the results of the leading order imaging series when low frequencies are missing. To implement this causality-like condition, the first reflector could be automatically picked or else, if we had a good estimate of the medium velocity down to the top of salt (Fig. 20), we could build that into the reference medium and begin the imaging series at a depth below the subsurface.

The importance of low frequency information to the accuracy of the leading order imaging series motivates the study of methods for spectral extrapolation (Walker and Ulrych, 1983; Innanen et al., 2004). Larger angles of incidence and the concomitant lower $k_{z}$ information is also expected to improve the accuracy of the depths predicted by the imaging series.

We note that trends in seismic data acquisition are towards the recording of lower frequencies. This is a welcome development as these and other new methods are developed that make full use of the recorded data's frequency spectrum.

## 6 Conclusion

We have demonstrated using reflectivity data in the precritical regime for a 1-D acoustic medium and 3-D wave propagation, that the leading order imaging series retains effectiveness, even when the input data are absent zero and low frequency information. This finding is critically important in the progression to a practical algorithm ready for application to field data. These conclusions merit the current generalization to a multidimensional earth (Liu et al., 2004) and more complex wave equations (Zhang and Weglein, 2004). The imaging of primaries at higher angles, especially in the post-critical regime is also of great interest (Nita and Weglein, 2004).

The imaging series is shown to benefit from having lower frequency information, in that reflectors will tend to move closer to their true depths than if the low frequency information is absent. Therefore, current trends towards acquiring lower frequency data, as well as existing and new methods for extrapolating this information are of keen interest.

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Figure 13: The effect of contrast between the actual and reference medium on the leading order imaging series for data missing low temporal frequency. A comparison of the derivatives with respect to $z$ of $\alpha_{1}$ (in red) and $\alpha^{\text {LOIM }}$ (in green) for $\theta_{0}=0$ and three different five-layer models. At top is the result for data that have all low frequencies, including zero frequency. In the other cases the low frequency limit is 4 Hz . The "middle contrast" model is the same as in Figs. 10-12 and the other two deviate from it by $\pm 50 \%$ in their interval velocities.


Figure 14: The variation of reflection coefficient at the first interface $\left(z=z_{a}\right)$ as a function of angle for two different models. The first term in the inverse series, $\alpha_{1}$, is proportional to $\cos ^{2} \theta_{0}$ times the data's amplitudes. The net result is that the amplitudes in $\alpha_{1}$ will tend to vary more gradually with angle than the amplitudes in the data.


Figure 15: The residual moveout for two different models corresponding to Fig. 14. $z_{b}$ is the actual depth of a reflector and $z_{b^{\prime}}$ is its depth predicted by the first term in the imaging series (i.e., through a conventional migration).


Figure 16: Filling in the low end of the $k_{z}$ spectrum according to (21) for 4-62.5 Hz bandlimited input data. A comparison, for four different angles of incidence, of $\alpha_{1}$, (in red) and $\alpha^{\text {LOIM (in green) and the } k_{z}}$ spectrum of $\alpha_{1}$ in each instance (on right).


Figure 17: A comparison of $\alpha_{1}$ (in red) and $\alpha^{\text {LOIM }}$ (in green) and the $k_{z}$ spectrum of $\alpha_{1}$ (on right) for four different low frequency limits. The filling in of the low end of the $k_{z}$ spectrum according to (21) is apparent. The difference in the depths predicted by the imaging series when having down to 0.125 Hz (top) and only having down to 2 Hz (second from top) is hardly noticeable.


Figure 18: Prestack leading order imaging series for a range of different low frequency limits. The left panel is the first term in the series, which exhibits typical residual moveout. The other three panels are the closed form result of the leading order imaging series. As more low frequency information is removed, the leading order imaging series results deteriorate but, in all cases, the results are an improvement over the first term.






Figure 20: The problem of subsalt imaging is sometimes described as a combination of complex, rugose top and bottom of salt and weak reflectivity below salt. The velocity down to the top of salt can often be adequately estimated using conventional velocity anlysis techniques.

# Target identification using the inverse scattering series: data requirements for the direct inversion of large-contrast, inhomogeneous elastic media 

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#### Abstract

In this paper we extend the earlier work (Zhang and Weglein, 2003) on direct non-linear inversion (parameter identification) of 1D acoustic media to a 1D isotropic inhomogeneous three parameter elastic medium. A formalism that generalizes the LS scalar acoustic equation to the elastic matrix case is presented and the associated inverse series provides a framework for direct elastic inverse processing. The first step in that process corresponds to direct linear inversion. An important new conclusion derived from this framework is that direct non-linear inversion for material properties of the simplest 1D elastic earth using line sources and receivers requires four components of data, $\left(\begin{array}{ll}\hat{D}^{P P} & \hat{D}^{P S} \\ \hat{D}^{S P} & \hat{D}^{S S}\end{array}\right)$, where the left subscripts of the matrices represent the type of measurement and the right ones are the source type. For example, $\hat{D}^{P S}$ is the P component of the measured scattered field corresponding to an incident S wave field in the reference medium. This provides an explanation of the often reported ambiguous results from indirect global search inversion of the $R_{P P}$ from the exact Zoeppritz's equations.


## 1 Introduction

In the context of exploration seismology, the promise of the inverse scattering series is to produce a direct non-linear inversion procedure in terms of data and reference information only. A strategy that has proven to be useful is to seek task isolated subseries contained within the series (Weglein et al., 2003) where these tasks are typically solved in sequence, from easiest to most difficult. These tasks typically associated with direct inversion are: (1) free surface multiple removal; (2) internal multiple removal; (3) imaging reflectors to their correct spatial location and (4) earth property identification. Determining subsurface material properties is the last and most difficult task to achieve. Weglein and Stolt (1992) introduce an elastic L-S equation and provide a specific linear inverse formalism for parameter estimation. Matson (1997) pioneered the development and application of methods for attenuating ocean bottom and on-shore multi component data.

The evolution of series-based methods from those associated with multiple removal to those associated with primaries (imaging at depth and parameter identification) began with 1D normal incidence acoustic analysis (Weglein et al., 2000, 2002; Shaw et al., 2003) to prestack 1D (Shaw and Weglein, 2004) and to prestack 1D two parameter acoustic (Zhang and Weglein, 2003) and 2D one parameter acoustic (Liu et al., 2004). Those analyses were able to isolate the imaging-only and inversion-only tasks within the series using explicit expressions for the inverse series in terms of their respective model parameters.

The objective of this paper is to begin similar analysis, i.e. towards a task (4) isolated subseries for a 1D elastic isotropic medium. The willingness of the inverse series to cooperate with our interests in task isolation is a non-trivial matter and significant effort (and luck) often is required in choosing favorable parameters and degrees of freedom to realize a map between our interests and the ways the series operates. We illustrate that issue for the acoustic two parameter model, where a slight change in parameters can make a significant difference in the ability to isolate tasks and to provide an interpretation to the terms of a given order in the inverse series for a given choice of parameters. Those lessons are invaluable in our ongoing efforts to choose parameters for the elastic inverse problem, that are most agreeable to physical interpretation in terms of imaging and inversion tasks.

However, overarching all considerations of physical interpretation and task specific identification is an unambiguous message from direct elastic inversion: for any choice of material property parameters, and choice of free parameter, the first step towards improvement beyond linear estimation of parameters requires the full data matrix $\left(\begin{array}{ll}\hat{D}^{P P} & \hat{D}^{P S} \\ \hat{D}^{S P} & \hat{D}^{S S}\end{array}\right)$. Strategies when only $\hat{D}^{P P}$ is available will be the subject of future reports.

In the following sections, at first, we give the background of elastic inversion in the displacement space ${ }^{1}$, then, we transform those operators and equations from the displacement space to the PS space. This is followed by the linear solution in PS space. For non-linear inversion, two results corresponding to two different sets of parameters are compared.

## 2 2D elastic inversion

In this section we consider the inversion problem in two dimensions for an elastic medium.

### 2.1 In the displacement space

We begin with some basic equations in the displacement space (Matson, 1997):

$$
\begin{equation*}
L \mathbf{u}=\mathbf{f} \tag{1}
\end{equation*}
$$

[^0]\[

$$
\begin{gather*}
L_{0} \mathbf{u}=\mathbf{f}  \tag{2}\\
L G=\mathbf{I}  \tag{3}\\
L_{0} G_{0}=\mathbf{I} \tag{4}
\end{gather*}
$$
\]

where $L$ and $L_{0}$ are the differential operators that describe the wave propagation in the actual and reference medium, respectively, $\mathbf{u}$ and $\mathbf{f}$ are the corresponding displacement and source terms, respectively, and $G$ and $G_{0}$ are the corresponding Green operators for the actual and reference medium.

Defining the perturbation $V=L_{0}-L$, the Lippmann- Schwinger equation for the elastic media in the displacement space is

$$
\begin{equation*}
G=G_{0}+G_{0} V G \tag{5}
\end{equation*}
$$

Iterating this equation back into itself generates the Born series

$$
\begin{equation*}
G=G_{0}+G_{0} V G_{0}+G_{0} V G_{0} V G_{0}+\cdots \tag{6}
\end{equation*}
$$

We define the data $D$ as the measured values of the scattered wave field. Then, on the measurement surface, we have

$$
\begin{equation*}
D=G_{0} V G_{0}+G_{0} V G_{0} V G_{0}+\cdots \tag{7}
\end{equation*}
$$

Expanding $V$ as a series in orders of $D$ (Weglein et al., 1997), we have

$$
\begin{equation*}
V=V_{1}+V_{2}+V_{3}+\cdots . \tag{8}
\end{equation*}
$$

Substituting Eq. (8) into Eq. (7), evaluating Eq. (7), and setting terms of equal order in the data equal, we get the equations that determine $V_{1}, V_{2}, \ldots$ from $D$ and $G_{0}$.

$$
\begin{gather*}
D=G_{0} V_{1} G_{0},  \tag{9}\\
0=G_{0} V_{2} G_{0}+G_{0} V_{1} G_{0} V_{1} G_{0}, \tag{10}
\end{gather*}
$$

In the actual medium, the 2-D elastic wave equation is (Weglein and Stolt, 1992)

$$
L \mathbf{u} \equiv\left[\rho \omega^{2}\left(\begin{array}{ll}
1 & 0  \tag{11}\\
0 & 1
\end{array}\right)+\left(\begin{array}{cc}
\partial_{1} \gamma \partial_{1}+\partial_{2} \mu \partial_{2} & \partial_{1}(\gamma-2 \mu) \partial_{2}+\partial_{2} \mu \partial_{1} \\
\partial_{2}(\gamma-2 \mu) \partial_{1}+\partial_{1} \mu \partial_{2} & \partial_{2} \gamma \partial_{2}+\partial_{1} \mu \partial_{1}
\end{array}\right)\right]\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right]=\mathbf{f}
$$

where
$\mathbf{u}=\left[\begin{array}{l}u_{1} \\ u_{2}\end{array}\right]=$ displacement,
$\rho=$ density,
$\gamma=$ bulk modulus $\left(\equiv \rho \alpha^{2}\right.$ where $\alpha=\mathrm{P}$ velocity),
$\mu=$ shear modulus ( $\equiv \rho \beta^{2}$ where $\beta=\mathrm{S}$ velocity),
$\omega=$ temporal frequency (angular), and
$\mathbf{f}$ is the source term.
For constant $(\rho, \gamma, \mu)=\left(\rho_{0}, \gamma_{0}, \mu_{0}\right),(\alpha, \beta)=\left(\alpha_{0}, \beta_{0}\right)$, the operator $L$ becomes

$$
L_{0} \equiv\left[\rho_{0} \omega^{2}\left(\begin{array}{ll}
1 & 0  \tag{12}\\
0 & 1
\end{array}\right)+\left(\begin{array}{cc}
\gamma_{0} \partial_{1}^{2}+\mu_{0} \partial_{2}^{2} & \left(\gamma_{0}-\mu_{0}\right) \partial_{1} \partial_{2} \\
\left(\gamma_{0}-\mu_{0}\right) \partial_{1} \partial_{2} & \mu_{0} \partial_{1}^{2}+\gamma_{0} \partial_{2}^{2}
\end{array}\right)\right] .
$$

Then,

$$
\begin{align*}
V & \equiv L_{0}-L \\
& =-\rho_{0}\left[\begin{array}{cc}
a_{\rho} \omega^{2}+\alpha_{0}^{2} \partial_{1} a_{\gamma} \partial_{1}+\beta_{0}^{2} \partial_{2} a_{\mu} \partial_{2} & \partial_{1}\left(\alpha_{0}^{2} a_{\gamma}-2 \beta_{0}^{2} a_{\mu}\right) \partial_{2}+\beta_{0}^{2} \partial_{2} a_{\mu} \partial_{1} \\
\partial_{2}\left(\alpha_{0}^{2} a_{\gamma}-2 \beta_{0}^{2} a_{\mu}\right) \partial_{1}+\beta_{0}^{2} \partial_{1} a_{\mu} \partial_{2} & a_{\rho} \omega^{2}+\alpha_{0}^{2} \partial_{2} a_{\gamma} \partial_{2}+\beta_{0}^{2} \partial_{1} a_{\mu} \partial_{1}
\end{array}\right], \tag{13}
\end{align*}
$$

where $a_{\rho} \equiv \frac{\rho}{\rho_{0}}-1, a_{\gamma} \equiv \frac{\gamma}{\gamma_{0}}-1$ and $a_{\mu} \equiv \frac{\mu}{\mu_{0}}-1$. For a 1D earth (i.e. $a_{\rho}, a_{\gamma}$ and $a_{\mu}$ are only functions of depth $z$ ), we have

$$
\left[\begin{array}{cc}
V^{11} & V^{12}  \tag{14}\\
V^{21} & V^{22}
\end{array}\right]=-\rho_{0}\left[\begin{array}{cc}
a_{\rho} \omega^{2}+\alpha_{0}^{2} a_{\gamma} \partial_{1}^{2}+\beta_{0}^{2} \partial_{2} a_{\mu} \partial_{2} & \left(\alpha_{0}^{2} a_{\gamma}-2 \beta_{0}^{2} a_{\mu}\right) \partial_{1} \partial_{2}+\beta_{0}^{2} \partial_{2} a_{\mu} \partial_{1} \\
\partial_{2}\left(\alpha_{0}^{2} a_{\gamma}-2 \beta_{0}^{2} a_{\mu}\right) \partial_{1}+\beta_{0}^{2} a_{\mu} \partial_{1} \partial_{2} & a_{\rho} \omega^{2}+\alpha_{0}^{2} \partial_{2} a_{\gamma} \partial_{2}+\beta_{0}^{2} a_{\mu} \partial_{1}^{2}
\end{array}\right] .
$$

### 2.2 Transform to PS space

In the reference medium, we diagonalize the operator $L_{0}$. Consider the transform matrix $\Pi=\left(\begin{array}{cc}\partial_{1} & \partial_{2} \\ -\partial_{2} & \partial_{1}\end{array}\right)$ and a constant matrix $\Gamma_{0}=\left(\begin{array}{cc}\gamma_{0} & 0 \\ 0 & \mu_{0}\end{array}\right)$, which satisfy

$$
\hat{L}_{0} \equiv \Pi L_{0} \Pi^{-1} \Gamma_{0}^{-1}=\left(\begin{array}{cc}
\hat{L}_{0}^{P} & 0 \\
0 & \hat{L}_{0}^{S}
\end{array}\right)
$$

where $\hat{L}_{0}$ is $L_{0}$ transformed to PS space, $\Pi^{-1}=\nabla^{-2}\left(\begin{array}{cc}\partial_{1} & -\partial_{2} \\ \partial_{2} & \partial_{1}\end{array}\right)$ is the inverse matrix of $\Pi$, $\hat{L}_{0}^{P}=\omega^{2} / \alpha_{0}^{2}+\nabla^{2}, \hat{L}_{0}^{S}=\omega^{2} / \beta_{0}^{2}+\nabla^{2}$. Multiplying Eq. (2) from the left by the operator $\Pi$, we find

$$
\begin{equation*}
\Pi L_{0} \Pi^{-1} \Gamma_{0}^{-1} \Gamma_{0} \Pi \mathbf{u}=\Pi \mathbf{f} \tag{15}
\end{equation*}
$$

Introduce as new independent variables the pressure $\phi_{P}$ and the shear stress $\phi_{S}$ defined as

$$
\boldsymbol{\Phi}=\binom{\phi_{P}}{\phi_{S}}=\Gamma_{0} \Pi \mathbf{u}=\left[\begin{array}{c}
\gamma_{0}\left(\partial_{1} u_{1}+\partial_{2} u_{2}\right)  \tag{16}\\
\mu_{0}\left(\partial_{1} u_{2}-\partial_{2} u_{1}\right)
\end{array}\right]
$$

and define

$$
\begin{equation*}
\mathbf{F}=\Pi \mathbf{f}=\binom{F^{P}}{F^{S}} . \tag{17}
\end{equation*}
$$

Then, in PS domain, Eq. (2) becomes,

$$
\left(\begin{array}{cc}
\hat{L}_{0}^{P} & 0  \tag{18}\\
0 & \hat{L}_{0}^{S}
\end{array}\right)\binom{\phi_{P}}{\phi_{S}}=\binom{F^{P}}{F^{S}} .
$$

Since $G_{0} \equiv L_{0}^{-1}$, let $\hat{G}_{0}^{P}=\left(\hat{L}_{0}^{P}\right)^{-1}$ and $\hat{G}_{0}^{S}=\left(\hat{L}_{0}^{S}\right)^{-1}$, then, displacement $G_{0}$ in PS domain becomes

$$
\hat{G}_{0}=\Gamma_{0} \Pi G_{0} \Pi^{-1}=\left(\begin{array}{cc}
\hat{G}_{0}^{P} & 0  \tag{19}\\
0 & \hat{G}_{0}^{S}
\end{array}\right) .
$$

So, in the reference medium, after transforming from the displacement domain to PS domain, both $L_{0}$ and $G_{0}$ become diagonal.
Multiplying Eq. (5) from the left by the operator $\Gamma_{0} \Pi$ and from the right by the operator $\Pi^{-1}$, and using Eq. (19),

$$
\begin{align*}
\Gamma_{0} \Pi G \Pi^{-1} & =\hat{G}_{0}+\hat{G}_{0}\left(\Pi V \Pi^{-1} \Gamma_{0}^{-1}\right) \Gamma_{0} \Pi G \Pi^{-1} \\
& =\hat{G}_{0}+\hat{G}_{0} \hat{V} \hat{G}, \tag{20}
\end{align*}
$$

where the displacement Green's operator $G$ is transformed to the PS domain as

$$
\hat{G}=\Gamma_{0} \Pi G \Pi^{-1}=\left(\begin{array}{cc}
\hat{G}^{P P} & \hat{G}^{P S}  \tag{21}\\
\hat{G}^{S P} & \hat{G}^{S S}
\end{array}\right) .
$$

The perturbation $V$ in the PS domain becomes

$$
\hat{V}=\Pi V \Pi^{-1} \Gamma_{0}^{-1}=\left(\begin{array}{ll}
\hat{V}^{P P} & \hat{V}^{P S}  \tag{22}\\
\hat{V}^{S P} & \hat{V}^{S S}
\end{array}\right),
$$

where, as before, the left subscripts of the matrices represent the type of measurement and the right ones are the source type.

Similarly, applying the PS transformation to the entire inverse series gives

$$
\begin{equation*}
\hat{V}=\hat{V}_{1}+\hat{V}_{2}+\hat{V}_{3}+\cdots . \tag{23}
\end{equation*}
$$

It follows, from Eqs. (20) and (23) that

$$
\begin{gather*}
\hat{D}=\hat{G}_{0} \hat{V}_{1} \hat{G}_{0},  \tag{24}\\
\hat{G}_{0} \hat{V}_{2} \hat{G}_{0}=-\hat{G}_{0} \hat{V}_{1} \hat{G}_{0} \hat{V}_{1} \hat{G}_{0}, \tag{25}
\end{gather*}
$$

where $\hat{D}=\left(\begin{array}{ll}\hat{D}^{P P} & \hat{D}^{P S} \\ \hat{D}^{S P} & \hat{D}^{S S}\end{array}\right)$ are the data in the PS domain.
In the displacement space we have, for Eq. (1),

$$
\begin{equation*}
\mathbf{u}=G \mathbf{f} \tag{26}
\end{equation*}
$$

Then, in the PS domain, Eq. (26) becomes

$$
\begin{equation*}
\Phi=\hat{G} \mathbf{F} \tag{27}
\end{equation*}
$$

On the measurement surface, we have

$$
\begin{equation*}
\hat{G}=\hat{G}_{0}+\hat{G}_{0} \hat{V}_{1} \hat{G}_{0} . \tag{28}
\end{equation*}
$$

We substitute Eq. (28) into Eq. (27), and rewrite Eq. (27) in matrix form:

$$
\binom{\phi_{P}}{\phi_{S}}=\left(\begin{array}{cc}
\hat{G}_{0}^{P} & 0  \tag{29}\\
0 & \hat{G}_{0}^{S}
\end{array}\right)\binom{F^{P}}{F^{S}}+\left(\begin{array}{cc}
\hat{G}_{0}^{P} & 0 \\
0 & \hat{G}_{0}^{S}
\end{array}\right)\left(\begin{array}{cc}
\hat{V}_{1}^{P P} & \hat{V}_{1}^{P S} \\
\hat{V}_{1}^{S P} & \hat{V}_{1}^{S S}
\end{array}\right)\left(\begin{array}{cc}
\hat{G}_{0}^{P} & 0 \\
0 & \hat{G}_{0}^{S}
\end{array}\right)\binom{F^{P}}{F^{S}}
$$

This can be written as the following two equations

$$
\begin{gather*}
\phi_{P}=\hat{G}_{0}^{P} F^{P}+\hat{G}_{0}^{P} \hat{V}_{1}^{P P} \hat{G}_{0}^{P} F^{P}+\hat{G}_{0}^{P} \hat{V}_{1}^{P S} \hat{G}_{0}^{S} F^{S},  \tag{30}\\
\phi_{S}=\hat{G}_{0}^{S} F^{S}+\hat{G}_{0}^{S} \hat{V}_{1}^{S P} \hat{G}_{0}^{P} F^{P}+\hat{G}_{0}^{S} \hat{V}_{1}^{S S} \hat{G}_{0}^{S} F^{S} \tag{31}
\end{gather*}
$$

We can see from the two equations above that for homogeneous media, (no perturbation, $\hat{V}_{1}=0$ ), there are only direct $P$ and $S$ waves and that the two kinds of waves are separated. However, for inhomogeneous media, these two kinds of waves will be mixed together. If only the P wave is incident, $F^{P}=1, F^{S}=0$, then the above two equations (30) and (31) are respectively reduced to

$$
\begin{gather*}
\phi_{P}=\hat{G}_{0}^{P}+\hat{G}_{0}^{P} \hat{V}_{1}^{P P} \hat{G}_{0}^{P}  \tag{32}\\
\phi_{S}=\hat{G}_{0}^{S} \hat{V}_{1}^{S P} \hat{G}_{0}^{P} \tag{33}
\end{gather*}
$$

Hence, in this case, there is only the direct P wave $\hat{G}_{0}^{P}$, and no direct wave S . But there are two kinds of scattered waves: one is the P-to-P wave $\hat{G}_{0}^{P} \hat{V}_{1}^{P P} \hat{G}_{0}^{P}$, and the other is the P-to-S wave $\hat{G}_{0}^{S} \hat{V}_{1}^{S P} \hat{G}_{0}^{P}$. For the acoustic case, only the P wave exists, and hence we only have one equation $\phi_{P}=\hat{G}_{0}^{P}+\hat{G}_{0}^{P} \hat{V}_{1}^{P P} \hat{G}_{0}^{P}$.
Similarly, if only the S wave is incident, $F^{P}=0, F^{S}=1$, and the two equations (30) and (31) are respectively reduced to

$$
\begin{gather*}
\phi_{P}=\hat{G}_{0}^{P} \hat{V}_{1}^{P S} \hat{G}_{0}^{S}  \tag{34}\\
\phi_{S}=\hat{G}_{0}^{S}+\hat{G}_{0}^{S} \hat{V}_{1}^{S S} \hat{G}_{0}^{S}, \tag{35}
\end{gather*}
$$

In this case, there is only the direct S wave $\hat{G}_{0}^{S}$, and no direct wave P . There are also two kinds of scattered waves: one is the S-to-P wave $\hat{G}_{0}^{P} \hat{V}_{1}^{P S} \hat{G}_{0}^{S}$, the other is the S-to-S wave $\hat{G}_{0}^{S} \hat{V}_{1}^{S S} \hat{G}_{0}^{S}$.

## 3 Linear inversion

Writing Eq. (24) in matrix form

$$
\left(\begin{array}{cc}
\hat{D}^{P P} & \hat{D}^{P S}  \tag{36}\\
\hat{D}^{S P} & \hat{D}^{S S}
\end{array}\right)=\left(\begin{array}{cc}
\hat{G}_{0}^{P} & 0 \\
0 & \hat{G}_{0}^{S}
\end{array}\right)\left(\begin{array}{cc}
\hat{V}_{1}^{P P} & \hat{V}_{1}^{P S} \\
\hat{V}_{1}^{S P} & \hat{V}_{1}^{S S}
\end{array}\right)\left(\begin{array}{cc}
\hat{G}_{0}^{P} & 0 \\
0 & \hat{G}_{0}^{S}
\end{array}\right),
$$

leads to four equations

$$
\begin{align*}
& \hat{D}^{P P}=\hat{G}_{0}^{P} \hat{V}_{1}^{P P} \hat{G}_{0}^{P},  \tag{37}\\
& \hat{D}^{P S}=\hat{G}_{0}^{P} \hat{V}_{1}^{P S} \hat{G}_{0}^{S},  \tag{38}\\
& \hat{D}^{S P}=\hat{G}_{0}^{S} \hat{V}_{1}^{S P} \hat{G}_{0}^{P},  \tag{39}\\
& \hat{D}^{S S}=\hat{G}_{0}^{S} \hat{V}_{1}^{S S} \hat{G}_{0}^{S} . \tag{40}
\end{align*}
$$

For $z_{s}=z_{g}=0$, in the $\left(k_{s}, z_{s} ; k_{g}, z_{g} ; \omega\right)$ domain, we get

$$
\begin{equation*}
\widetilde{D}^{P P}\left(k_{g}, 0 ;-k_{g}, 0 ; \omega\right)=-\frac{1}{4}\left(1-\frac{k_{g}^{2}}{\nu_{g}^{2}}\right) \widetilde{a}_{\rho}\left(-2 \nu_{g}\right)-\frac{1}{4}\left(1+\frac{k_{g}^{2}}{\nu_{g}^{2}}\right) \widetilde{a}_{\gamma}\left(-2 \nu_{g}\right)+\frac{2 k_{g}^{2} \beta_{0}^{2}}{\left(\nu_{g}^{2}+k_{g}^{2}\right) \alpha_{0}^{2}} \widetilde{a}_{\mu}\left(-2 \nu_{g}\right), \tag{41}
\end{equation*}
$$

$\widetilde{D}^{P S}\left(\nu_{g}, \eta_{g}\right)=-\frac{1}{4}\left(\frac{k_{g}}{\nu_{g}}+\frac{k_{g}}{\eta_{g}}\right) \widetilde{a}_{\rho}\left(-\nu_{g}-\eta_{g}\right)-\frac{\beta_{0}^{2}}{2 \omega^{2}} k_{g}\left(\nu_{g}+\eta_{g}\right)\left(1-\frac{k_{g}^{2}}{\nu_{g} \eta_{g}}\right) \widetilde{a}_{\mu}\left(-\nu_{g}-\eta_{g}\right)$,
$\widetilde{D}^{S P}\left(\nu_{g}, \eta_{g}\right)=\frac{1}{4}\left(\frac{k_{g}}{\nu_{g}}+\frac{k_{g}}{\eta_{g}}\right) \widetilde{a}_{\rho}\left(-\nu_{g}-\eta_{g}\right)+\frac{\beta_{0}^{2}}{2 \omega^{2}} k_{g}\left(\nu_{g}+\eta_{g}\right)\left(1-\frac{k_{g}^{2}}{\nu_{g} \eta_{g}}\right) \widetilde{a}_{\mu}\left(-\nu_{g}-\eta_{g}\right)$,

$$
\begin{equation*}
\widetilde{D}^{S S}\left(k_{g}, \eta_{g}\right)=-\frac{1}{4}\left(1-\frac{k_{g}^{2}}{\eta_{g}^{2}}\right) \widetilde{a}_{\rho}\left(-2 \eta_{g}\right)-\left[\frac{\eta_{g}^{2}+k_{g}^{2}}{4 \eta_{g}^{2}}-\frac{2 k_{g}^{2}}{\eta_{g}^{2}+k_{g}^{2}}\right] \widetilde{a}_{\mu}\left(-2 \eta_{g}\right), \tag{43}
\end{equation*}
$$

where

$$
\begin{aligned}
\nu_{g}^{2}+k_{g}^{2} & =\frac{\omega^{2}}{\alpha_{0}^{2}} \\
\eta_{g}^{2}+k_{g}^{2} & =\frac{\omega^{2}}{\beta_{0}^{2}}
\end{aligned}
$$

## 4 Non-linear inversion

Writing Eq. (25) in matrix form:

$$
\begin{align*}
& \left(\begin{array}{cc}
\hat{G}_{0}^{P} & 0 \\
0 & \hat{G}_{0}^{S}
\end{array}\right)\left(\begin{array}{cc}
\hat{V}_{2}^{P P} & \hat{V}_{2}^{P S} \\
\hat{V}_{2}^{S P} & \hat{V}_{2}^{S S}
\end{array}\right)\left(\begin{array}{cc}
\hat{G}_{0}^{P} & 0 \\
0 & \hat{G}_{0}^{S}
\end{array}\right) \\
& =-\left(\begin{array}{cc}
\hat{G}_{0}^{P} & 0 \\
0 & \hat{G}_{0}^{S}
\end{array}\right)\left(\begin{array}{cc}
\hat{V}_{1}^{P P} & \hat{V}_{1}^{P S} \\
\hat{V}_{1}^{S P} & \hat{V}_{1}^{S S}
\end{array}\right)\left(\begin{array}{cc}
\hat{G}_{0}^{P} & 0 \\
0 & \hat{G}_{0}^{S}
\end{array}\right)\left(\begin{array}{cc}
\hat{V}_{1}^{P P} & \hat{V}_{1}^{P S} \\
\hat{V}_{1}^{S P} & \hat{V}_{1}^{S S}
\end{array}\right)\left(\begin{array}{cc}
\hat{G}_{0}^{P} & 0 \\
0 & \hat{G}_{0}^{S}
\end{array}\right), \tag{45}
\end{align*}
$$

leads to four equations

$$
\begin{align*}
\hat{G}_{0}^{P} \hat{V}_{2}^{P P} \hat{G}_{0}^{P} & =-\hat{G}_{0}^{P} \hat{V}_{1}^{P P} \hat{G}_{0}^{P} \hat{V}_{1}^{P P} \hat{G}_{0}^{P}-\hat{G}_{0}^{P} \hat{V}_{1}^{P S} \hat{G}_{0}^{S} \hat{V}_{1}^{S P} \hat{G}_{0}^{P}  \tag{46}\\
\hat{G}_{0}^{P} \hat{V}_{2}^{P S} \hat{G}_{0}^{S} & =-\hat{G}_{0}^{P} \hat{V}_{1}^{P P} \hat{G}_{0}^{P} \hat{V}_{1}^{P S} \hat{G}_{0}^{S}-\hat{G}_{0}^{P} \hat{V}_{1}^{P S} \hat{G}_{0}^{S} \hat{V}_{1}^{S S} \hat{G}_{0}^{S}  \tag{47}\\
\hat{G}_{0}^{S} \hat{V}_{2}^{S P} \hat{G}_{0}^{P} & =-\hat{G}_{0}^{S} \hat{V}_{1}^{S P} \hat{G}_{0}^{P} \hat{V}_{1}^{P P} \hat{G}_{0}^{P}-\hat{G}_{0}^{S} \hat{V}_{1}^{S S} \hat{G}_{0}^{S} \hat{V}_{1}^{S P} \hat{G}_{0}^{P}  \tag{48}\\
\hat{G}_{0}^{S} \hat{V}_{0}^{S S} \hat{G}_{0}^{S} & =\hat{G}_{0}^{S} \hat{V}_{1}^{S P} \hat{G}_{0}^{P} \hat{V}_{1}^{P S} \hat{G}_{0}^{S}-\hat{G}_{0}^{S} \hat{V}_{1}^{S S} \hat{G}_{0}^{S} \hat{V}_{1}^{S S} \hat{G}_{0}^{S} \tag{49}
\end{align*}
$$

From the equations above, we can see that we cannot perform the direct non-linear inversion without knowing all components of the data.

In the derivation for the second order inversion, some difficulties were encountered which made us think about the parametrization in inversion. The choice of the parameters and the definition of the perturbation for inversion turns out to be a crucial step.

In the acoustic case, if we start directly with the pressure wave equation and choose $\theta$ as the free parameter, $\alpha$ and $\beta$ as the two material property parameters, (Weglein et al., 2003; Zhang and Weglein, 2003) we arrive at the following equation for the second order (first term beyond linear):

$$
\begin{align*}
& \frac{1}{\cos ^{2} \theta} \alpha_{2}(z)+\left(1-\tan ^{2} \theta\right) \beta_{2}(z) \\
= & -\frac{1}{2 \cos ^{4} \theta} \alpha_{1}^{2}(z) \\
& -\frac{1}{2}\left(1+\tan ^{4} \theta\right) \beta_{1}^{2}(z) \\
& +\frac{\tan ^{2} \theta}{\cos ^{2} \theta} \alpha_{1}(z) \beta_{1}(z) \\
& -\frac{1}{2 \cos ^{4} \theta} \alpha_{1}^{\prime}(z) \int_{0}^{z} d z^{\prime}\left[\alpha_{1}\left(z^{\prime}\right)-\beta_{1}\left(z^{\prime}\right)\right] \\
& +\frac{1}{2}\left(\tan ^{4} \theta-1\right) \beta_{1}^{\prime}(z) \int_{0}^{z} d z^{\prime}\left[\alpha_{1}\left(z^{\prime}\right)-\beta_{1}\left(z^{\prime}\right)\right] . \tag{50}
\end{align*}
$$

If we start with the displacement domain, as discussed, letting $\mu_{0}, \beta_{0}, \mu$, and $\beta=0$ and choose $\theta$ as the free parameter, $a_{\gamma}$ and $a_{\rho}$ as the two material property parameters, the following solution for second order is produced:

$$
\begin{align*}
& \frac{1}{\cos ^{2} \theta} a_{\gamma}^{(2)}(z)+\left(1-\tan ^{2} \theta\right) a_{\rho}^{(2)}(z) \\
= & -\frac{1}{2}\left(\tan ^{4} \theta-1\right) a_{\gamma}^{(1)^{2}}(z) \\
& -\frac{1}{2}\left(\frac{1}{\cos ^{4} \theta}-2\right) a_{\rho}^{(1)^{2}}(z) \\
& +\frac{\tan ^{2} \theta}{\cos ^{2} \theta} a_{\gamma}^{(1)}(z) a_{\rho}^{(1)}(z) \\
& -\frac{1}{2 \cos ^{4} \theta} a_{\gamma}^{(1)^{\prime}}(z) \int_{0}^{z} d z^{\prime}\left[a_{\gamma}^{(1)}\left(z^{\prime}\right)-a_{\rho}^{(1)}\left(z^{\prime}\right)\right] \\
& +\frac{1}{2}\left(\tan ^{4} \theta-1\right) a_{\rho}^{(1)^{\prime}}(z) \int_{0}^{z} d z^{\prime}\left[a_{\gamma}^{(1)}\left(z^{\prime}\right)-a_{\rho}^{(1)}\left(z^{\prime}\right)\right] \\
& +A(z) \tag{51}
\end{align*}
$$

where the definition of $\theta$ is the same as that of Eq. (50), $a_{\gamma}^{(1)^{\prime}}=\frac{d a_{\gamma}^{(1)}}{d z}, a_{\rho}^{(1)^{\prime}}=\frac{d a_{\rho}^{(1)}}{d z}$, and $A(z)$ in $\left(\nu_{g}\right)$ domain, i.e., before the Fourier transform over $\nu_{g}$, is

$$
\begin{aligned}
\widetilde{A}\left(\nu_{g}\right)= & \frac{1}{i \pi \nu_{g}} \int_{-\infty}^{+\infty} d z^{\prime} a_{\rho}^{(1)^{\prime}}\left(z^{\prime}\right) a_{\gamma}^{(1)}\left(z^{\prime}\right) e^{2 i \nu_{g} z^{\prime}} \\
+ & \frac{\tan \theta}{\pi} \frac{1}{\nu_{g}} \int_{-\infty}^{+\infty} d z^{\prime} \int_{-\infty}^{+\infty} d z^{\prime \prime} a_{\rho}^{(1)^{\prime}}\left(z^{\prime}\right) a_{\rho}^{(1)^{\prime}}\left(z^{\prime \prime}\right) H\left(z^{\prime}-z^{\prime \prime}\right) e^{i \nu_{g}(1+i \tan \theta) z^{\prime}} e^{i \nu_{g}(1-i \tan \theta) z^{\prime \prime}} \\
+ & \frac{1}{2 \pi \cos ^{2} \theta} \frac{1}{i \nu_{g}(1-i \tan \theta)} \cdot \\
& \cdot \int_{-\infty}^{+\infty} d z^{\prime} \int_{-\infty}^{+\infty} d z^{\prime \prime} a_{\rho}^{(1)^{\prime}}\left(z^{\prime}\right) a_{\gamma}^{(1)^{\prime}}\left(z^{\prime \prime}\right) H\left(z^{\prime}-z^{\prime \prime}\right) e^{i \nu_{g}(1+i \tan \theta) z^{\prime}} e^{i \nu_{g}(1-i \tan \theta) z^{\prime \prime}} \\
- & \frac{1}{2 \pi \cos ^{2} \theta} \frac{1}{i \nu_{g}(1+i \tan \theta)} \cdot \\
& \cdot \int_{-\infty}^{+\infty} d z^{\prime} \int_{-\infty}^{+\infty} d z^{\prime \prime} a_{\rho}^{(1)^{\prime}}\left(z^{\prime}\right) a_{\gamma}^{(1)^{\prime}}\left(z^{\prime \prime}\right) H\left(z^{\prime \prime}-z^{\prime}\right) e^{i \nu_{g}(1-i \tan \theta) z^{\prime}} e^{i \nu_{g}(1+i \tan \theta) z^{\prime \prime}} \\
& +\frac{1}{2 \pi} \frac{1}{\nu_{g}^{2}} \int_{-\infty}^{+\infty} d z^{\prime} \int_{-\infty}^{+\infty} d z^{\prime \prime}\left\{e^{i \nu_{g} z^{\prime}} a_{\rho}^{(1)^{\prime}}\left(z^{\prime}\right) \operatorname{sgn}\left(z^{\prime}-z^{\prime \prime}\right)\left[e^{i \nu_{g}\left|z^{\prime}-z^{\prime \prime}\right|}-e^{-k_{g}\left|z^{\prime}-z^{\prime \prime}\right|}\right]\right. \\
& \left.\cdot\left[a_{\gamma}^{(1)^{\prime \prime}}\left(z^{\prime \prime}\right)+2 i \nu_{g} a_{\gamma}^{(1)^{\prime}}\left(z^{\prime \prime}\right)\right] e^{i \nu_{g} z^{\prime \prime}}\right\} .
\end{aligned}
$$

Obviously, the second method or solution is much more complex than the first one. In addition, if another choice of free parameter other than $\theta$ (e.g., $\omega$ or $k_{h}$ ) was selected, then the functional form between the data and the first order perturbation changes. Furthermore, the relationship between the first and second order perturbation is then also different, and new analysis will then be required for the purposes of identifying specific task separated subseries. In our experience, the choice of $\theta$ as free parameter (for a 1D medium) is particularly well suited for allowing a task separated identification of terms in the inverse series.

For acoustic media with fixed density, from Eq. (50), we have

$$
\begin{equation*}
\alpha_{2}(z)=-\frac{1}{2 \cos ^{2} \theta}\left[\alpha_{1}^{2}(z)+\alpha_{1}^{\prime}(z) \int_{0}^{z} d z^{\prime} \alpha_{1}\left(z^{\prime}\right)\right] \tag{52}
\end{equation*}
$$

while from Eq. (51) we have

$$
\begin{equation*}
a_{\gamma}^{(2)}(z)=-\frac{1}{2}\left(\tan ^{2} \theta-1\right) a_{\gamma}^{(1)^{2}}(z)-\frac{1}{2 \cos ^{2} \theta} a_{\gamma}^{(1)^{\prime}}(z) \int_{0}^{z} d z^{\prime} a_{\gamma}^{(1)}\left(z^{\prime}\right) \tag{53}
\end{equation*}
$$

Since $\alpha=1-\frac{\gamma_{0}}{\gamma}$, then

$$
a_{\gamma}=\frac{\gamma}{\gamma_{0}}-1=\frac{\alpha}{1-\alpha}=\alpha+\alpha^{2}+\alpha^{3}+\cdots,
$$

where the series expansion is valid for $|\alpha|<1$. And then we have

$$
\begin{gathered}
a_{\gamma}^{(1)}=\alpha_{1}, \\
a_{\gamma}^{(2)}=\alpha_{2}+\alpha_{1}^{2},
\end{gathered}
$$

Then,

$$
a_{\gamma}^{(2)}=-\frac{1}{2 \cos ^{2} \theta}\left[\alpha_{1}^{2}(z)+\alpha_{1}^{\prime}(z) \int_{0}^{z} d z^{\prime} \alpha_{1}\left(z^{\prime}\right)\right]+\alpha_{1}^{2}(z)
$$

from the equation above we can get Eq. (53):

$$
a_{\gamma}^{(2)}(z)=-\frac{1}{2}\left(\tan ^{2} \theta-1\right) a_{\gamma}^{(1)^{2}}(z)-\frac{1}{2 \cos ^{2} \theta} a_{\gamma}^{(1)^{\prime}}(z) \int_{0}^{z} d z^{\prime} a_{\gamma}^{(1)}\left(z^{\prime}\right)
$$

Eq. (52) and Eq. (53) agree!
Next, for acoustic media with variable density, from Eq. (50) we have

$$
\begin{equation*}
\left(1-\tan ^{2} \theta\right) \beta_{2}(z)=-\frac{1}{2}\left(1+\tan ^{4} \theta\right) \beta_{1}^{2}(z)-\frac{1}{2}\left(\tan ^{4} \theta-1\right) \beta_{1}^{\prime}(z) \int_{0}^{z} d z^{\prime} \beta_{1}\left(z^{\prime}\right), \tag{54}
\end{equation*}
$$

while from Eq. (51) we can get

$$
\begin{equation*}
\left(1-\tan ^{2} \theta\right) a_{\rho}^{(2)}(z)=-\frac{1}{2}\left(\frac{1}{\cos ^{4} \theta}-2\right) a_{\rho}^{(1)^{2}}(z)-\frac{1}{2}\left(\tan ^{4} \theta-1\right) a_{\rho}^{(1)^{\prime}}(z) \int_{0}^{z} d z^{\prime} a_{\rho}^{(1)}\left(z^{\prime}\right)+B(z) \tag{55}
\end{equation*}
$$

where $B(z)$ in $\left(\nu_{g}\right)$ domain, i.e., before the Fourier transform over $\nu_{g}$, is

$$
\widetilde{B}\left(\nu_{g}\right)=\frac{\tan \theta}{\pi} \frac{1}{\nu_{g}} \int_{-\infty}^{+\infty} d z^{\prime} \int_{-\infty}^{+\infty} d z^{\prime \prime} a_{\rho}^{(1)^{\prime}}\left(z^{\prime}\right) a_{\rho}^{(1)^{\prime}}\left(z^{\prime \prime}\right) H\left(z^{\prime}-z^{\prime \prime}\right) e^{i \nu_{g}(1+i \tan \theta) z^{\prime}} e^{i \nu_{g}(1-i \tan \theta) z^{\prime \prime}}
$$

Since $\beta=1-\frac{\rho_{0}}{\rho}$, then

$$
a_{\rho}=\frac{\rho}{\rho_{0}}-1=\frac{\beta}{1-\beta}=\beta+\beta^{2}+\beta^{3}+\cdots,
$$

where the series expansion is valid for $|\beta|<1$. And then we have

$$
\begin{gathered}
a_{\rho}^{(1)}=\beta_{1}, \\
a_{\rho}^{(2)}=\beta_{2}+\beta_{1}^{2},
\end{gathered}
$$

Then,

$$
\begin{aligned}
\left(1-\tan ^{2} \theta\right) a_{\rho}^{(2)}(z)= & {\left[-\frac{1}{2}\left(1+\tan ^{4} \theta\right) \beta_{1}^{2}(z)-\frac{1}{2}\left(\tan ^{4} \theta-1\right) \beta_{1}^{\prime}(z) \int_{0}^{z} d z^{\prime} \beta_{1}\left(z^{\prime}\right)\right] } \\
& +\left(1-\tan ^{2} \theta\right) \beta_{1}^{2}(z)
\end{aligned}
$$

from the equation above we can get

$$
\left(1-\tan ^{2} \theta\right) a_{\rho}^{(2)}(z)=-\frac{1}{2}\left(\frac{1}{\cos ^{4} \theta}-2\right) a_{\rho}^{(1)^{2}}(z)-\frac{1}{2}\left(\tan ^{4} \theta-1\right) a_{\rho}^{(1)^{\prime}}(z) \int_{0}^{z} d z^{\prime} a_{\rho}^{(1)}\left(z^{\prime}\right)
$$

The amplitude term(s) and the imaging term(s) in Eq. (54) and Eq. (55) agree!

## 5 Conclusion

This paper provides a framework and analysis of issues involved in data requirements, computation and interpretation of the non-linear direct elastic inverse problem. Specifically, a detailed analysis of how different choice of acoustic parameters (and free parameters) have a marked difference on the ability of task separated interpretation. That analysis provides a guide and lesson for ongoing effort at parameter inversion and structural location specific subseries for the elastic world.

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# Inverse scattering series for laterally-varying media 

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#### Abstract

We consider the extension of the current casting of the inverse scattering series for imaging and inverting primary seismic energy (Weglein et al., 2002) to cases involving lateral variation in the medium parameters. The extension is developed theoretically to the second term in the series, and a form is presented which is interpretable in the same way as previously-derived terms in the absence of lateral variation. We present a preliminary numerical result that shows lateral and vertical correction of location and amplitude to occur, in accordance with current understanding of second order imaging and inversion terms/numerics. These results are very encouraging, and represent the first concrete numerical indication of multidimensional imaging in the absence of a correct velocity model.


## 1 Introduction

The inverse scattering series is the only known theory permitting the multidimensional direct inversion of seismic wave field measurements. To date (Weglein et al., 2003; Shaw et al., 2003; Zhang and Weglein, 2003) focus has been on the detailed understanding of the operations associated with subseries' for imaging (reflector location) and inversion (target identification) in 1D, for both normal incidence and 1D-with-offset cases (and various acoustic and elastic models). We have also considered the analogous development of terms given a variable reference medium (Liu and Weglein, 2002). Encouraging numerical results in the case of data with missing zero-frequency in 1D (Shaw and Weglein, 2004) and/or methods for extrapolation to low- and zero-frequency (Innanen et al., 2004) have been found and/or developed. But the true power of these imaging and inversion methods is in their ability to handle multi-dimensional media. In this paper we approach this issue, by developing and testing these subseries' for media which vary in lateral as well as vertical coordinates.

We make conscious choices in this derivation to produce terms which mirror those of the 1D cases previously investigated. At an intermediate point in the development, we separate into two alternate derivations, one that is more efficient for a numerical implementation, the other being better suited to interpretation in a task-specific framework. We further note:
(1) The development is expressed in the midpoint-conjugate/depth domain, i.e. $\left(k_{m}, z\right)$.
(2) We consider a medium which varies in $P$-wave velocity only, i.e. we assume constant density everywhere.
(3) The extra degree of freedom occurring in this single-parameter development is fixed by choosing offset-conjugate $k_{h}=0$ in the numerical development.

## 2 Theoretical Development

We begin by reviewing the first and second order portions of the inverse scattering series, closely following the development of, e.g., Weglein et al. (2003). The desired scattering potential $V$, which for our purposes describes perturbations of wavespeed away from an at most slowly-varying background wavespeed $c_{0}(x, z)$ :

$$
\begin{equation*}
V(x, z, \omega)=\frac{\omega^{2}}{c_{0}^{2}(x, y)} \alpha(x, z) \tag{1}
\end{equation*}
$$

(in which $\alpha(x, z)=1-c_{0}^{2}(x, z) / c^{2}(x, z)$ for a true wavespeed distribution $c(x, z)$ ), has the linear component

$$
\begin{equation*}
V_{1}(x, z, \omega)=\frac{\omega^{2}}{c_{0}^{2}(x, z)} \alpha_{1}(x, z) \tag{2}
\end{equation*}
$$

and second order component

$$
\begin{equation*}
V_{2}(x, z, \omega)=\frac{\omega^{2}}{c_{0}^{2}(x, z)} \alpha_{2}(x, z) \tag{3}
\end{equation*}
$$

The linear portion of the inverse scattering series is an exact relationship between $V_{1}$ and the scattered field evaluated on a measurement surface (i.e. the data $D$ ). In operator form, this relationship is

$$
\begin{equation*}
\mathrm{G}_{0} \mathrm{~V}_{1} \mathrm{G}_{0}=\mathrm{D}=\mathrm{G}-\mathrm{G}_{0} \tag{4}
\end{equation*}
$$

Causal Green's functions for homogeneous media are given by (following Clayton and Stolt, 1981):

$$
\begin{equation*}
G_{0}\left(k_{g}, z_{g}, x^{\prime}, z^{\prime}, \omega\right)=\frac{\rho_{r}}{2 i} \frac{e^{-i k_{g} x^{\prime}} e^{i q_{g}\left|z_{g}-z^{\prime}\right|}}{q_{g}} \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
G_{0}\left(x^{\prime}, z^{\prime}, k_{s}, z_{s}, \omega\right)=\frac{\rho_{r}}{2 i} \frac{e^{i k_{s} x^{\prime}} e^{i q_{s}\left|z_{s}-z^{\prime}\right|}}{q_{s}} \tag{6}
\end{equation*}
$$

where

$$
\begin{equation*}
q_{g}=\operatorname{sgn}(\omega) \sqrt{\left(\frac{\omega}{c_{0}}\right)^{2}-k_{g}^{2}} \tag{7}
\end{equation*}
$$

and

$$
\begin{equation*}
q_{s}=\operatorname{sgn}(\omega) \sqrt{\left(\frac{\omega}{c_{0}}\right)^{2}-k_{s}^{2}} \tag{8}
\end{equation*}
$$

In the above expressions, $k_{g}$ is the Fourier conjugate to $x_{g}$, and likewise $k_{s}$ is conjugate to $x_{s}$. Note that the "sign convention" of the Fourier transform is different for the source and geophone coordinates. This is a convenient choice in the present formalism.

Equation (4) may be solved by applying a Fourier transform over the lateral source and receiver coordinates to obtain $\alpha_{1}$. Here we consider the solution of the second-order inverse scattering series equations:

$$
\begin{equation*}
G_{0} V_{2} G_{0}=-G_{0} V_{1} G_{0} V_{1} G_{0} \tag{9}
\end{equation*}
$$

Expressing equation (9) explicitly given our choice of $V_{2}$, we have

$$
\begin{align*}
& \int_{-\infty}^{\infty} d x^{\prime} \int_{-\infty}^{\infty} d z^{\prime} G_{0}\left(x_{g}, z_{g}, x^{\prime}, z^{\prime}, \omega\right) V_{2}\left(x^{\prime}, z^{\prime}, \omega\right) G_{0}\left(x^{\prime}, z^{\prime}, x_{s}, z_{s}, \omega\right) \\
& =-\int_{-\infty}^{\infty} d x^{\prime} \int_{-\infty}^{\infty} d z^{\prime} G_{0}\left(x_{g}, z_{g}, x^{\prime}, z^{\prime}, \omega\right) V_{1}\left(x^{\prime}, z^{\prime}, \omega\right)  \tag{10}\\
& \times \int_{-\infty}^{\infty} d x^{\prime \prime} \int_{-\infty}^{\infty} d z^{\prime \prime} G_{0}\left(x^{\prime}, z^{\prime}, x^{\prime \prime}, z^{\prime \prime}, \omega\right) V_{1}\left(x^{\prime \prime}, z^{\prime \prime}, \omega\right) G_{0}\left(x^{\prime \prime}, z^{\prime \prime}, k_{s}, z_{s}, \omega\right) .
\end{align*}
$$

We next Fourier transform over lateral geophone and shot coordinates: $\int_{-\infty}^{\infty} d x_{g} \int_{-\infty}^{\infty} d x_{s} e^{i k_{s} x_{s}-i k_{g} x_{g}}$, and express the internal Green's function as

$$
\begin{equation*}
G_{0}\left(x^{\prime}, z^{\prime}, x^{\prime \prime}, z^{\prime \prime}, \omega\right)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} d k_{\lambda} e^{i k_{\lambda} x^{\prime}} G_{0}\left(k_{\lambda}, z^{\prime}, x^{\prime \prime}, z^{\prime \prime}, \omega\right) \tag{11}
\end{equation*}
$$

(in which $k_{\lambda}$ is conjugate to $x^{\prime}$ ). This results in

$$
\begin{align*}
-\frac{\rho_{r}}{4 c_{0}^{2}} \frac{\omega^{2}}{q_{g} q_{s}} \widetilde{\widetilde{\alpha}}_{2}\left(k_{g}-k_{s}, k_{z}\right)= & -\frac{i \rho_{r}}{16 \pi c_{0}^{4}} \int_{-\infty}^{\infty} d k_{\lambda} \frac{\omega^{4}}{q_{g} q_{\lambda} q_{s}} \int_{-\infty}^{\infty} d z^{\prime} \widetilde{\alpha}_{1}\left(k_{g}-k_{\lambda}, z^{\prime}\right)  \tag{12}\\
& \times \int_{-\infty}^{\infty} d z^{\prime \prime} \widetilde{\alpha}_{1}\left(k_{\lambda}-k_{s}, z^{\prime \prime}\right) e^{i\left[q_{g}\left(z^{\prime}-z_{g}\right)+q_{\lambda}\left|z^{\prime \prime}-z^{\prime}\right|+q_{s}\left(z^{\prime \prime}-z_{s}\right)\right]}
\end{align*}
$$

where $k_{z}=\sqrt{\omega^{2} / c_{0}^{2}-k_{g}^{2}}+\sqrt{\omega^{2} / c_{0}^{2}-k_{s}^{2}}$ is the vertical wavenumber. The quantity $\widetilde{\alpha}_{1}\left(k_{m}, z\right)$ is the Fourier transform of $\alpha_{1}\left(x_{m}, z\right)$, and $\widetilde{\alpha}_{1}\left(k_{m}, k_{z}\right)$ is the Fourier transform over both $x_{m}$ and $z$. For computational reasons and to allow an easy comparison with 1D acoustic earth, we will compute $\alpha_{2}$ in the $\left(k_{m}, z\right)$ domain.

With this in mind, we apply the inverse Fourier transform $(1 / 2 \pi) \int_{-\infty}^{\infty} e^{i k_{z} z} d k_{z}$ to equation (12), resulting in:

$$
\begin{align*}
\widetilde{\alpha}_{2}\left(k_{g}-k_{s}, z\right)= & \frac{1}{8 \pi^{2} c_{0}^{2}} \int_{-\infty}^{\infty} d k_{\lambda} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d z^{\prime \prime} \int_{-\infty}^{\infty} d k_{z} \\
& \frac{i \omega^{2}}{q_{\lambda}} \widetilde{\alpha}_{1}\left(k_{g}-k_{\lambda}, z^{\prime}\right) \widetilde{\alpha}_{1}\left(k_{\lambda}-k_{s}, z^{\prime \prime}\right) e^{i\left[q_{g}\left(z^{\prime}-z\right)+q_{\lambda}\left|z^{\prime \prime}-z^{\prime}\right|+q_{s}\left(z^{\prime \prime}-z\right)\right]} . \tag{13}
\end{align*}
$$

This expression can be shown to reduce to the 1D form of $\alpha_{2}(z)$ discussed and analyzed elsewhere. See Appendix C for a detailed account of this reduction.
From this point (equation 13) we may proceed with development in two ways; one way, shown in the next section, produces quantities that are more convenient for numerical implementations. The section after that summarizes a development that more closely mirrors the task-separation strategy adopted in previous simpler cases (i.e. $\alpha_{1}(z)$ ).

## 3 Derivation I: Numerically Implemented Form

The innermost integral of equation (13) contains $\widetilde{\alpha}_{1}$, which depends on the measurement of the wave field; it can be taken out of this integral (with respect to $k_{z}$ ) if it can be shown that $k_{g}-k_{\lambda}$ and $k_{\lambda}-k_{s}$ are not a function of $k_{z}$. This can be accomplished in many ways, for instance by fixing the Fourier conjugate $x_{h}$ of the lateral offset coordinate to be constant. For convenience, we choose that constant to be $k_{h}=0$. (See Clayton and Stolt, 1981 for a more detailed discussion.) Making this choice, we have:

$$
\begin{align*}
k_{h} & =k_{g}+k_{s}=0 \\
\frac{\omega}{c_{0}} & =\frac{1}{2} \operatorname{sgn}\left(k_{z}\right) \sqrt{k_{z}^{2}+k_{m}^{2}}  \tag{14}\\
k_{g} & =-k_{s}=\frac{1}{2} k_{m}
\end{align*}
$$

which results in a simplified expression for $\widetilde{\alpha}_{2}\left(k_{m}, z\right)$ :

$$
\begin{align*}
\widetilde{\alpha}_{2}\left(k_{m}, z\right)= & \frac{1}{16 \pi^{2}} \int_{-\infty}^{\infty} d k_{\lambda} \int_{0}^{\infty} d z^{\prime} \widetilde{\alpha}_{1}\left(0.5 k_{m}-k_{\lambda}, z^{\prime}\right) \\
& \times\left\{\int_{0}^{\infty} d z^{\prime \prime} \widetilde{\alpha}_{1}\left(k_{\lambda}+0.5 k_{m}, z^{\prime \prime}\right)\right\} \gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right), \tag{15}
\end{align*}
$$

where

$$
\begin{equation*}
\gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right)=i \int_{-\infty}^{\infty} d k_{z} \frac{k_{z}^{2}+k_{m}^{2}}{\operatorname{sgn}\left(k_{z}\right) \sqrt{k_{z}^{2}+k_{m}^{2}-4 k_{\lambda}^{2}}} e^{i\left[z_{1} k_{z}+z_{2} \operatorname{sgn}\left(k_{z}\right) \sqrt{k_{z}^{2}+k_{m}^{2}-4 k_{\lambda}^{2}}\right]} \tag{16}
\end{equation*}
$$

and

$$
\begin{align*}
& z_{1}=0.5\left(z^{\prime}+z^{\prime \prime}\right)-z,  \tag{17}\\
& z_{2}=0.5\left|z^{\prime}-z^{\prime \prime}\right| .
\end{align*}
$$

Notice that the expression $\gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right)$ does not depend on the measured data; we may compute it once and use it repeatedly, saving on computation. The computation is further simplified by taking advantage of the symmetries:

$$
\begin{align*}
& \gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right) \\
= & \gamma\left(z, z^{\prime}, z^{\prime \prime},-k_{\lambda}\right) \\
= & \gamma\left(z, z^{\prime \prime}, z^{\prime}, k_{\lambda}\right)  \tag{18}\\
= & \gamma\left(z, z^{\prime \prime}, z^{\prime},-k_{\lambda}\right),
\end{align*}
$$

which reduces the integration interval by half:

$$
\begin{align*}
\widetilde{\alpha}_{2}\left(k_{m}, z\right)= & \frac{1}{8 \pi^{2}} \int_{-\infty}^{\infty} d k_{\lambda} \int_{0}^{\infty} d z^{\prime} \widetilde{\alpha}_{1}\left(0.5 k_{m}-k_{\lambda}, z^{\prime}\right) \\
& \times\left\{\int_{0}^{z^{\prime}} d z^{\prime \prime} \widetilde{\alpha}_{1}\left(k_{\lambda}+0.5 k_{m}, z^{\prime \prime}\right)\right\} \gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right) . \tag{19}
\end{align*}
$$

### 3.1 An Evaluation of the Integral $\gamma$

We consider the evaluation of the integral $\gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right)$ as seen in the foregoing derivation of the second order perturbation. We define: $\left|k_{m}^{2}-4 k_{\lambda}^{2}\right|=a^{2}$. The quantity $\widetilde{\alpha}_{2}\left(k_{m}, z\right)$ may be written

$$
\begin{equation*}
\widetilde{\alpha}_{2}\left(k_{m}, z\right)=\frac{1}{8 \pi^{2}}\left(I_{1}+I_{12}+I_{22}+I_{\mathrm{pre}}+I_{\mathrm{post}}+I_{\mathrm{sc}}\right) \tag{20}
\end{equation*}
$$

where

$$
\begin{align*}
I_{1}= & -2 \pi \int_{-\infty}^{\infty} d k_{\lambda} \widetilde{\alpha}_{1}\left(0.5 k_{m}-k_{\lambda}, z\right) \widetilde{\alpha}_{1}\left(k_{\lambda}+0.5 k_{m}, z\right) \\
& -2 \pi \int_{-\infty}^{\infty} d k_{\lambda} \frac{\partial \widetilde{\alpha}_{1}\left(0.5 k_{m}-k_{\lambda}, z\right)}{\partial z} \int_{0}^{z} d z^{\prime} \widetilde{\alpha}_{1}\left(k_{\lambda}+0.5 k_{m}, z^{\prime}\right), \tag{21}
\end{align*}
$$

$$
\begin{align*}
I_{21}= & -2 \int_{-0.5 k_{m}}^{0.5 k_{m}} d k_{\lambda} \int_{-\infty}^{\infty} d z^{\prime} \widetilde{\alpha}_{1}\left(0.5 k_{m}-k_{\lambda}, z^{\prime}\right) \int_{0}^{z} d z^{\prime \prime} \widetilde{\alpha}_{1}\left(k_{\lambda}+0.5 k_{m}, z^{\prime \prime}\right) \\
& \times \frac{0.5 a\left(z^{\prime}-z^{\prime \prime}\right) \cos \left(0.5 a\left(z^{\prime}-z^{\prime \prime}\right)\right)-\sin \left(0.5 a\left(z^{\prime}-z^{\prime \prime}\right)\right)}{\left(z-z^{\prime}\right)^{2}} \tag{22}
\end{align*}
$$

$$
\begin{align*}
I_{22}= & -2 \int_{\left|k_{\lambda}\right|>0.5 k_{m}} d k_{\lambda} \int_{-\infty}^{\infty} d z^{\prime} \widetilde{\alpha}_{1}\left(0.5 k_{m}-k_{\lambda}, z^{\prime}\right) \int_{0}^{z} d z^{\prime \prime} \widetilde{\alpha}_{1}\left(k_{\lambda}+0.5 k_{m}, z^{\prime \prime}\right)  \tag{23}\\
& \times \frac{0.5 a\left(z^{\prime}+z^{\prime \prime}-2 z\right) \cos \left(0.5 a\left(z^{\prime}+z^{\prime \prime}-2 z\right)\right)-\sin \left(0.5 a\left(z^{\prime}+z^{\prime \prime}-2 z\right)\right)}{\left(z-z^{\prime}\right)^{2}}
\end{align*}
$$

$$
\begin{align*}
I_{\text {post }}= & -2 \pi \int_{\left|k_{\lambda}\right|>0.5 k_{m}} d k_{\lambda} \int_{-\infty}^{\infty} d z^{\prime} \operatorname{sgn}\left(z^{\prime}-z\right) \widetilde{\alpha}_{1}\left(0.5 k_{m}-k_{\lambda}, z^{\prime}\right) \int_{-\infty}^{z^{\prime}} d z^{\prime \prime} \widetilde{\alpha}_{1}\left(k_{\lambda}+0.5 k_{m}, z^{\prime \prime}\right) \\
& \times\left\{\frac{a^{2} z_{f}}{\left|z_{e}\right|} f_{32}\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right)-\frac{4 k_{\lambda}^{2} z_{2}^{2}-k_{m}^{2} z_{1}^{2}}{z_{e}} f_{42}\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right)\right\}, \tag{25}
\end{align*}
$$

and

$$
\begin{align*}
I_{\text {sc }}= & 2 \pi i \times \int_{\left|k_{\lambda}\right|>0.5 k_{m}} d k_{\lambda} \int_{-\infty}^{\infty} d z^{\prime} \widetilde{\alpha}_{1}\left(0.5 k_{m}-k_{\lambda}, z^{\prime}\right) \int_{-\infty}^{z^{\prime}} d z^{\prime \prime} \widetilde{\alpha}_{1}\left(k_{\lambda}+0.5 k_{m}, z^{\prime \prime}\right)  \tag{26}\\
& \times\left\{4 k_{\lambda}^{2} H_{J 2}\left(a z_{1}, \text { fluc }=a z_{2}\right)+k_{m}^{2} H_{J 3}\left(a z_{1}, \text { fluc }=a z_{2}\right)\right\} .
\end{align*}
$$

The functions $f$ in the above expressions are given by

$$
\begin{align*}
f_{31}\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right) & =H\left(z_{e}\right) H_{k 2}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\frac{\operatorname{sgn}\left(z^{\prime}-z\right) z_{2}}{\sqrt{\left|z_{e}\right|}}\right) \\
& +H\left(-z_{e}\right) H_{N 2}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\frac{\operatorname{sgn}\left(z^{\prime}-z\right) z_{2}}{\sqrt{\left|z_{e}\right|}}\right) \\
f_{32}\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right) & =H\left(z_{e}\right) H_{N 2}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\frac{\operatorname{sgn}\left(z^{\prime}-z\right) z_{1}}{\sqrt{\left|z_{e}\right|}}\right) \\
& +H\left(-z_{e}\right) H_{k 2}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\frac{\operatorname{sgn}\left(z^{\prime}-z\right) z_{1}}{\sqrt{\left|z_{e}\right|}}\right)  \tag{27}\\
f_{41}\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right) & =H\left(z_{e}\right) H_{k}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\frac{\operatorname{sgn}\left(z^{\prime}-z\right) z_{2}}{\sqrt{\left|z_{e}\right|}}\right) \\
& +H\left(-z_{e}\right) H_{N}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\frac{\operatorname{sgn}\left(z^{\prime}-z\right) z_{2}}{\sqrt{\left|z_{e}\right|}}\right) \\
f_{42}\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right) & =H\left(z_{e}\right) H_{N}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\frac{\operatorname{sgn}\left(z^{\prime}-z\right) z_{1}}{\sqrt{\left|z_{e}\right|}}\right) \\
& +H\left(-z_{e}\right) H_{k}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\frac{\operatorname{sgn}\left(z^{\prime}-z\right) z_{1}}{\sqrt{\left|z_{e}\right|}}\right)
\end{align*}
$$

also

$$
\begin{align*}
& z_{e}=z_{1}^{2}-z_{2}^{2}  \tag{28}\\
& z_{f}=z_{1}^{2}+z_{2}^{2}
\end{align*}
$$

The functions $H$ are given in Appendix B. In Appendix A we provide the remaining definitions of variables in the expressions for $I_{1}, I_{21}, I_{22}, I_{\text {pre }}, I_{\text {post }}$, and $I_{\mathrm{sc}}$ above, and develop the strategy for their derivation.

Consider $I_{1}$ first. Notice that if the $\gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right)$ integral is approximated by its leading order Taylor's series term of $\gamma\left(q_{\lambda}\right)$ calculated at $q_{\lambda}=q$, then the expression for $\widetilde{\alpha}_{2}\left(k_{m}, z\right)$ reduces to this term only. The rest of the terms can hence be interpreted as compensating for this approximation.

## 4 Derivation II: Task-Separated Form

The development thus far has been for an amalgamated second order perturbation, in the sense that we have not identified separable terms that appear to concern themselves only
with either imaging or inversion tasks. In this section we demonstrate the separation of the second-order relationship given by equation (19) to reflect the 1D separated form, i.e. into terms that are related exclusively to the tasks of imaging and inversion (c.f. Weglein et al., 2003).

Consider the formula in the $\left(k_{m}, k_{z}\right)$ domain; we follow an "integration by parts logic" similar to that done for the 1D case and as reported in previously referenced literature. We may write:

$$
\begin{align*}
\widetilde{\widetilde{\alpha}}_{2}\left(k_{m}, k_{z}\right) & =\frac{k_{0}^{2}}{4 \pi i} \int_{-\infty}^{\infty} d k_{\lambda} \frac{1}{q_{\lambda}} \\
& \times\left\{\int_{-\infty}^{\infty} d z^{\prime} e^{i z^{\prime}\left(q_{s}+q_{g}\right)}\left(\frac{1}{i\left(q_{g}+q_{\lambda}\right)}+\frac{1}{i\left(q_{s}+q_{\lambda}\right)}\right) \widetilde{\alpha}_{1}\left(k_{g}-k_{\lambda}, z^{\prime}\right) \widetilde{\alpha}_{1}\left(k_{\lambda}-k_{s}, z^{\prime}\right)\right. \\
& +\int_{-\infty}^{\infty} d z^{\prime} \frac{e^{i z^{\prime}\left(q_{g}+q_{\lambda}\right)}}{i\left(q_{g}+q_{\lambda}\right)} \widetilde{\alpha}_{1}^{\prime}\left(k_{g}-k_{\lambda}, z^{\prime}\right) \int_{-\infty}^{z^{\prime}} d z^{\prime \prime} e^{i z^{\prime \prime}\left(q_{g}-q_{\lambda}\right)} \widetilde{\alpha}_{1}\left(k_{\lambda}-k_{s}, z^{\prime \prime}\right) \\
& \left.+\int_{-\infty}^{\infty} d z^{\prime} \frac{e^{i z^{\prime}\left(q_{s}+q_{\lambda}\right)}}{i\left(q_{s}+q_{\lambda}\right)} \widetilde{\alpha}_{1}^{\prime}\left(k_{\lambda}-k_{s}, z^{\prime}\right) \int_{-\infty}^{z^{\prime}} d z^{\prime \prime} e^{i z^{\prime \prime}\left(q_{g}-q_{\lambda}\right)} \widetilde{\alpha}_{1}\left(k_{g}-k_{\lambda}, z^{\prime \prime}\right)\right\}, \tag{29}
\end{align*}
$$

where $k_{m}=k_{g}-k_{s}$ and $k_{z}=-q_{g}-q_{s}$. The topmost term has a form that resembles that of the self-interaction type term in the 1D case, i.e. in which the multiplicatively/nonlinearly contributing portions of the linear perturbation $\alpha_{1}$ are constrained to be collocated. The remaining two terms likewise greatly resemble the imaging portion of the 1D case, i.e. they involve the first derivative in depth of the linear perturbation whose amplitude is modified by the first integral of the same linear perturbation.
For $k_{h}=0$, the choice that was made in the derivation for the numerical computations, we obtain

$$
\begin{align*}
\widetilde{\widetilde{\alpha}}_{2}\left(k_{m}, k_{z}\right)= & -\frac{k_{0}^{2}}{2 \pi} \int_{-\infty}^{\infty} d k_{\lambda} \frac{1}{q_{\lambda}\left(q+q_{\lambda}\right)}\left[\int_{-\infty}^{\infty} d z^{\prime} e^{i z^{\prime} 2 q_{9}} \widetilde{\alpha}_{1}\left(k_{m} / 2-k_{\lambda} z^{\prime}\right) \widetilde{\alpha}_{1}\left(k_{m} / 2+k_{\lambda} z^{\prime}\right)\right. \\
& \left.+\int_{-\infty}^{\infty} d z^{\prime} e^{i z^{\prime}\left(q+q_{\lambda}\right)} \widetilde{\alpha}_{1}^{\prime}\left(k_{m} / 2-k_{\lambda} z^{\prime}\right) \int_{-\infty}^{z^{\prime}} d z^{\prime \prime} e^{i z^{\prime \prime}\left(q-q_{\lambda}\right)} \widetilde{\alpha}_{1}\left(k_{\lambda}-k_{m} / 2, z^{\prime \prime}\right)\right] \tag{30}
\end{align*}
$$

where $q=q_{g}=q_{s}$. If we consider the leading order Taylor's series term of the coefficient and the phase only as functions of $q_{\lambda}$ calculated at $q_{\lambda}=q$ (as mentioned in our above discussion on the evaluation of $\gamma$ ), and then inverse Fourier transform on $k_{z}$ we find the approximate expression

$$
\begin{equation*}
\widetilde{\alpha}_{2}\left(k_{m}, z\right) \approx I_{1} \tag{31}
\end{equation*}
$$

In other words, the two derivations are consistent with one another.

## 5 Numerical Example

In this section we present the result of a preliminary numerical implementation of the second order inverse scattering series terms for the recovery of laterally-varying medium parameters. The geological model used is pictured in Figure 1. The model is laterally-invariant at its edges, but at its center we place an incline in the top interface. Below this is a horizontal reflector. The first interface is at $z=500 \mathrm{~m}$ depth on the left side, and at $z=300 \mathrm{~m}$ depth on the right; the lower interface is fixed at $z=700 \mathrm{~m}$. At the surface $(z=0 \mathrm{~m})$ sources and receivers are placed on a regular grid. (Pictured many receivers and a single shot at the center.)

Figure 2 illustrates a typical shot gather from synthetic data. The data is generated using the aforementioned model and a finite difference scheme, and utilizing the first derivative of a Gaussian function as the source wavelet.

Figure 3 illustrates the linear inverse result, i.e. $\alpha_{1}(x, z)$. The key is in the deflection of the lower interface away from horizontal, due to the laterally-varying overburden. This reflector, correctly imaged, would naturally lie closer to fully horizontal.


Figure 1: Model used in numerical test of second order inverse scattering series terms for laterally-varying media. Top interface varies in the lateral coordinate direction near the center of the model.


Figure 2: Typical shot gather computed using a finite difference numerical scheme. The source wavelet is the first derivative of a Gaussian.


Figure 3: The linear inverse results, i.e. $\alpha_{1}(x, z)$. Notice the deflection of the lower reflector from horizontal.


Figure 4: (a) The portion of $\alpha_{2}$ which is responsible for imaging, calculated using the second part of Eq.(21). The single strong event in this figure will move the second reflector towards its correct location. Notice the right half is stronger than the left; this half has experienced a thicker high velocity zone, so we need a stronger correction. There is no need to move the first event because its location is already correct. (b) Illustrated is the inversion, or amplitude altering, portion of $\alpha_{2}$.


Figure 5: Comparison of various terms near the edge of the left half. In red is the original $\alpha_{1}$, in green is $\alpha_{1}+\alpha_{2}$, in yellow is the imaging part of $\alpha_{2}$, in blue is the parameter inversion part of $\alpha_{2}$, and in black is the sum of the imaging and inversion part of $\alpha_{2}$. The dashed gray line shows the actual location of the first and second reflector. Both $\alpha_{1}$ and $\alpha_{1}+\alpha_{2}$ show the correct location of the first reflector, but $\alpha_{1}+\alpha_{2}$ successfully moves the second reflector towards its correct location.


Figure 6: Comparison of various terms near the edge of the left half. In red is the original $\alpha_{1}$, in green is $\alpha_{1}+\alpha_{2}$, in yellow is the imaging part of $\alpha_{2}$, in blue is the parameter inversion part of $\alpha_{2}$, and in black is the sum of the imaging and inversion part of $\alpha_{2}$. The dashed gray line shows the actual location of the first and second reflector. Both $\alpha_{1}$ and $\alpha_{1}+\alpha_{2}$ show the correct location of the first reflector, but $\alpha_{1}+\alpha_{2}$ successfully moves the second reflector towards its correct location. Compared with the previous figure, this portion of the model has a deeper high velocity zone, and the second reflector is further deflected up. The second term $\alpha_{2}$ has moved this part of the second reflector a greater distance as required. This encouraging result is our first numerical evidence of imaging in the presence of lateral medium variations without knowledge of the actual velocity field.


Figure 7: Detail around the central part of the second reflector. (To make the image more like seismic data, the derivative over $z$ is displayed.) (a) $\alpha_{1}+\alpha_{2}$ is shown; (b) $\alpha_{1}$ is shown. In the central part of the model, which has maximal lateral variation, the second term moves our target toward its correct location. To calculate how far the second reflector has moved, we automatically pick the location of the second reflector using a maximum energy criterion, in green is the desired location, in yellow is the location picked from the original $\alpha_{1}$, and in red is the location picked from $\alpha_{1}+\alpha_{2}$. To make the comparison easier, all 3 horizons are shown in both (a) and (b).

## 6 Conclusions

We present a formalism for the nonlinear imaging and inversion of 2D wave field data over a medium with laterally-varying parameters. The development is motivated and follows closely the co-development of 1D (depth-varying) methods. We consider the second order term in the series for the wavespeed perturbation $\alpha(x, z)$, namely $\alpha_{2}(x, z)$.
We further present a form of this second order algorithm that directly mirrors the purposeful casting of the terms as those which correspond separately and exclusively to tasks of imaging and inversion. These can be seen to compare closely to their 1D brethren.

Numerically we see encouraging results given an unknown overburden with lateral variation. This research represents the initial foray into the domains in which the inverse scattering series stands alone as a formalism for direct inversion seismic data: multiple dimensions. Future research, long and short term, involves the pursuit of more general physical models, greater dimensionality, and the inclusion of higher order terms.

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## Appendix A

In this appendix we illustrate the basic strategies behind the re-expression of $\gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right)$. We have

$$
\begin{equation*}
\gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right)=i \int_{-\infty}^{\infty} d k_{z} \frac{k_{z}^{2}+k_{m}^{2}}{\operatorname{sgn}\left(k_{z}\right) \sqrt{k_{z}^{2}+k_{m}^{2}-4 k_{\lambda}^{2}}} e^{i\left[z_{1} k_{z}+z_{2} \operatorname{sgn}\left(k_{z}\right) \sqrt{k_{z}^{2}+k_{m}^{2}-4 k_{\lambda}^{2}}\right]} \tag{32}
\end{equation*}
$$

where

$$
\begin{align*}
& z_{1}=0.5\left(z^{\prime}+z^{\prime \prime}\right)-z \\
& z_{2}=0.5\left|z^{\prime}-z^{\prime \prime}\right| . \tag{33}
\end{align*}
$$

We have seen that symmetry may be exploited to make computation more efficient. As a result, we may consider only specific domains of depth variables, in fact, we may drop the absolute value sign in the definition of $z_{2}$, such that $z_{2}=0.5\left(z^{\prime}-z^{\prime \prime}\right)$. Furthermore, depending on whether $0.5 k_{m}^{2}-k_{\lambda}^{2}>0$ or $0.5 k_{m}^{2}-k_{\lambda}^{2}<0$, we may define

$$
\begin{equation*}
\pm a^{2}=k_{m}^{2}-4 k_{\lambda}^{2}, \tag{34}
\end{equation*}
$$

such that

$$
\begin{equation*}
\gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right)=i \int_{-\infty}^{\infty} d k_{z} \frac{k_{z}^{2}+k_{m}^{2}}{\operatorname{sgn}\left(k_{z}\right) \sqrt{k_{z}^{2} \pm a^{2}}} e^{i\left[z_{1} k_{z}+z_{2} \operatorname{sgn}\left(k_{z}\right) \sqrt{k_{z}^{2} \pm a^{2}}\right]}, \tag{35}
\end{equation*}
$$

and with a further change of variables to $\kappa_{z}=k_{z} / a$, the integral becomes

$$
\begin{equation*}
\gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right)=i \int_{-\infty}^{\infty} d \kappa_{z} \frac{a^{2} \kappa_{z}^{2}+k_{m}^{2}}{\operatorname{sgn}\left(\kappa_{z}\right) \sqrt{\kappa_{z}^{2} \pm 1}} e^{i a\left[z_{1} \kappa_{z}+z_{2} \operatorname{sgn}\left(\kappa_{z}\right) \sqrt{\kappa_{z}^{2} \pm 1}\right]} \tag{36}
\end{equation*}
$$

The phase term in the above expression is more suitably expressed using a further variable change:

$$
\begin{align*}
& \phi=z_{1} \kappa_{z}+z_{2} \operatorname{sgn}\left(\kappa_{z}\right) \sqrt{\kappa_{z}^{2} \pm 1}, \\
& \psi=\operatorname{sgn}(\phi) \sqrt{\phi^{2} \mp\left(z_{2}^{2}-z_{1}^{2}\right) .} \tag{37}
\end{align*}
$$

Using these new variables, we may re-express many of the terms in the integral. To wit:

$$
\begin{align*}
& \kappa_{z}=\frac{-z_{1} \phi+z_{2} \psi}{z_{2}^{2}-z_{1}^{2}} \\
& \operatorname{sgn}\left(\kappa_{z}\right) \sqrt{\kappa_{z}^{2} \pm 1}=\frac{z_{2} \phi-z_{1} \psi}{z_{2}^{2}-z_{1}^{2}}, \\
& d \kappa_{z}=\frac{z_{2} \phi-z_{1} \psi}{\psi\left(z_{2}^{2}-z_{1}^{2}\right)} d \phi  \tag{38}\\
& i e^{i a \phi} \frac{a \kappa_{z}^{2}+k_{m}^{2}}{\operatorname{sgn}\left(\kappa_{z}\right) \sqrt{\kappa_{z}^{2} \pm 1}} \\
& =i e^{-i a \phi} d \phi\left\{\left[\frac{a}{z_{1}+z_{2}}\right]^{2} \phi+\frac{2 a^{2}\left(z_{1}^{2}+z_{2}^{2}\right)}{\left(z_{2}^{2}-z_{1}^{2}\right)^{2}} \frac{\phi(\phi-\psi)}{\psi}+\left[k_{m}^{2} \mp \frac{a^{2} z_{2}^{2}}{z_{2}^{2}-z_{1}^{2}}\right] \frac{1}{\psi}\right\} .
\end{align*}
$$

The strategy is to utilize this integral as a normal Fourier transform, where the variable is now $\phi$ rather than $k_{z}$; however, the integral range of $\phi$ behaves in a more complicated manner, and requires some case-by-case analysis. (We find it to be truncated in some cases, and integrated multiply in others.)
Consider the first case $0<k_{m}^{2}-4 k_{\lambda}^{2}=a^{2}$. This begets two subcases:
(1) $z_{1}+z_{2}>0$, in which (as $k_{z}$ goes from $-\infty$ to $\infty$ ) $\phi$ ranges from $-\infty$ to $-z_{2}$, then jumps to the interval $z_{2}$ to $\infty$.
(2) $z_{1}+z_{2}<0$, in which (again as $k_{z}$ goes from $-\infty$ to $\infty$ ) $\phi$ ranges from $\infty$ to $-z_{2}$, then jumps up to the interval $z_{2}$ to $-\infty$.

This integration range is summarized as follows:

$$
\begin{align*}
L_{1}+L_{2}+L_{3}+L_{4} & =\left(\operatorname{sgn}\left(z_{1}+z_{2}\right) \int_{-\infty}^{\infty}-\int_{-z_{2}}^{z_{2}}\right) i e^{-i a \phi} d \phi \\
& \times\left\{\left[\frac{a}{z_{1}+z_{2}}\right]^{2} \phi+\frac{a^{2}\left(z_{1}^{2}+z_{2}^{2}\right)}{\left(z_{2}^{2}-z_{1}^{2}\right)^{2}} \frac{\phi(\phi-\psi)}{\psi}+\left[k_{m}^{2}-\frac{a^{2} z_{2}^{2}}{z_{2}^{2}-z_{1}^{2}}\right] \frac{1}{\psi}\right\} . \tag{39}
\end{align*}
$$

The component $L_{1}$ being

$$
\begin{align*}
L_{1} & =\operatorname{sgn}\left(z_{1}+z_{2}\right) \int_{-\infty}^{\infty} d \phi i e^{-i a \phi}\left(\frac{a}{z_{1}+z_{2}}\right)^{2} \phi \\
& =\frac{\operatorname{sgn}\left(z_{1}+z_{2}\right)}{\left(z_{1}+z_{2}\right)^{2}} a^{2} \int_{-\infty}^{\infty} d \phi i \phi e^{-i a \phi}  \tag{40}\\
& =\frac{\operatorname{sgn}\left(z_{1}+z_{2}\right)}{\left(z_{1}+z_{2}\right)^{2}} a^{2} \delta^{\prime}(a) \\
& =\delta^{\prime}\left(z_{1}+z_{2}\right) .
\end{align*}
$$

The last step is justified by making use of the identity $\operatorname{sgn}(w) / w^{2} \delta^{\prime}(x)=\delta^{\prime}(w x)$, and the fact that $a$ is always positive. We thus get a sense of the efficiency associated with this way of expressing the integral: the sifting aspect of these delta-like quantities produces simple results.

The second integral $L_{2}$ is:

$$
\begin{align*}
L_{2} & =-\int_{-z_{2}}^{z_{2}} d \phi i e^{-i a \phi}\left(\frac{a}{z_{1}+z_{2}}\right)^{2} \phi \\
& =-\frac{2\left[a z_{2} \cos a z_{2}-\sin a z_{2}\right]}{\left(z_{1}+z_{2}\right)^{2}} . \tag{41}
\end{align*}
$$

The remaining parts of the integral are likewise computable. For example, for the third term $L_{3}$, we have:

$$
\begin{equation*}
L_{3}=\left(\operatorname{sgn}\left(z_{1}+z_{2}\right) \int_{-\infty}^{\infty}-\int_{-z_{2}}^{z_{2}}\right) i e^{-i a \phi} d \phi \frac{a^{2}\left(z_{1}^{2}+z_{2}^{2}\right)}{\left(z_{2}^{2}-z_{1}^{2}\right)^{2}} \frac{\phi(\phi-\psi)}{\psi} . \tag{42}
\end{equation*}
$$

The integral above can be simplified by making the following changes of variables: $z_{e}=$ $z_{1}^{2}-z_{2}^{2}, z_{f}=z_{1}^{2}+z_{2}^{2}, \nu=\frac{\phi}{\sqrt{\left|z_{e}\right|}}$, and notice the fact that in this case: $\psi=\operatorname{sgn} \sqrt{\phi^{2}-\left(z_{2}^{2}-z_{1}^{2}\right)}$, we have:

$$
\begin{equation*}
L_{3}=\left(\operatorname{sgn}\left(z_{1}+z_{2}\right) \int_{-\infty}^{\infty}-\int_{-\frac{z_{2}}{\sqrt{\left|z_{e}\right|}}}^{\frac{z_{2}}{\sqrt{\left|z_{e}\right|}}}\right) i e^{-i a \sqrt{\left|z_{e}\right|} \nu} d \nu \frac{a^{2} z_{f}}{\left|z_{e}\right|} \frac{\nu\left(\nu-\operatorname{sgn} \sqrt{\nu^{2} \pm 1}\right)}{\operatorname{sgn} \sqrt{\nu^{2} \pm 1}} . \tag{43}
\end{equation*}
$$

where the plus or minus sign in the equation above depends on whether $z_{1}^{2}-z_{2}^{2}>0$ or $z_{1}^{2}-z_{2}^{2}<0$. The integral above can be easily expressed by our pre-defined functions, in the case of $z_{2}^{2}<z_{1}^{2}$, we have

$$
\begin{equation*}
L_{3}=-2 \pi \frac{a^{2} z_{f}}{\left|z_{e}\right|}\left(\operatorname{sgn}\left(z_{1}+z_{2}\right)\right) \times H_{K 2}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\operatorname{sgn}\left(z_{1}+z_{2}\right) \frac{z_{2}}{\sqrt{\left|z_{e}\right|}}\right) \tag{44}
\end{equation*}
$$

In the case of $z_{2}^{2}>z_{1}^{2}$, we have

$$
\begin{equation*}
L_{3}=-2 \pi \frac{a^{2} z_{f}}{\left|z_{e}\right|}\left(\operatorname{sgn}\left(z_{1}+z_{2}\right)\right) \times H_{N 2}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\operatorname{sgn}\left(z_{1}+z_{2}\right) \frac{z_{2}}{\sqrt{\left|z_{e}\right|}}\right) \tag{45}
\end{equation*}
$$

For the fourth term $L_{4}$, we have:

$$
\begin{equation*}
L_{4}=\left(\operatorname{sgn}\left(z_{1}+z_{2}\right) \int_{-\infty}^{\infty}-\int_{-z_{2}}^{z_{2}}\right) i e^{-i a \phi} d \phi\left(k_{m}^{2}-\frac{a^{2} z_{2}^{2}}{z_{2}^{2}-z_{1}^{2}}\right) \frac{1}{\psi} \tag{46}
\end{equation*}
$$

Using the same change-of-variables used for $L_{3}$, we calculate $L_{4}$.

$$
\begin{equation*}
L_{4}=\left(\operatorname{sgn}\left(z_{1}+z_{2}\right) \int_{-\infty}^{\infty}-\int_{-\frac{z_{2}}{\sqrt{\left|z_{e}\right|}}}^{\frac{z_{2}}{\sqrt{\left|z_{e}\right|}}}\right) i e^{-i a \sqrt{\left|z_{e}\right| \nu}} d \nu\left(k_{m}^{2}-\frac{a^{2} z_{2}^{2}}{z_{2}^{2}-z_{1}^{2}}\right) \frac{1}{\sqrt{\nu^{2} \pm 1}} . \tag{47}
\end{equation*}
$$

where the plus or minus sign in the equation above depends on whether $z_{1}^{2}-z_{2}^{2}>0$ or $z_{1}^{2}-z_{2}^{2}<0$. The integral above is exactly the integral we defined in Appendix B, so, in the case of $z_{2}^{2}<z_{1}^{2}$ :

$$
\begin{equation*}
L_{4}=2 \pi \frac{4 k_{\lambda}^{2} z_{2}^{2}-k_{m}^{2} z_{1}^{2}}{z_{e}}\left(\operatorname{sgn}\left(z_{1}+z_{2}\right)\right) \times H_{K}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\operatorname{sgn}\left(z_{1}+z_{2}\right) \frac{z_{2}}{\sqrt{\left|z_{e}\right|}}\right) \tag{48}
\end{equation*}
$$

In the case of $z_{2}^{2}>z_{1}^{2}$, we have

$$
\begin{equation*}
L_{4}=2 \pi \frac{4 k_{\lambda}^{2} z_{2}^{2}-k_{m}^{2} z_{1}^{2}}{z_{e}}\left(\operatorname{sgn}\left(z_{1}+z_{2}\right)\right) \times H_{N}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\operatorname{sgn}\left(z_{1}+z_{2}\right) \frac{z_{2}}{\sqrt{\left|z_{e}\right|}}\right) \tag{49}
\end{equation*}
$$

Next, we consider the second case $0>k_{m}^{2}-4 k_{\lambda}^{2}=-a^{2}$. In this case, within the finite interval: $-a<k_{z}<a, \sqrt{k_{z}^{2}-a^{2}}$ will be imaginary. We introduce specifically defined functions to express this portion of integral:

$$
\begin{equation*}
L_{\mathrm{sc}}=i \int_{-a}^{a} d k_{z} \frac{k_{z}^{2}+k_{m}^{2}}{\operatorname{sgn}\left(k_{z}\right) \sqrt{k_{z}^{2}-a^{2}}} e^{i\left[z_{1} k_{z}+z_{2} \operatorname{sgn}\left(k_{z}\right) \sqrt{k_{z}^{2}-a^{2}}\right]} . \tag{50}
\end{equation*}
$$

After changing the integration variable: $\kappa_{z}=k_{z} / a$, we have:

$$
\begin{equation*}
L_{\mathrm{sc}}=i \int_{-1}^{1} d \kappa_{z} \frac{a^{2} \kappa_{z}^{2}+k_{m}^{2}}{\operatorname{sgn}\left(\kappa_{z}\right) \sqrt{\kappa_{z}^{2}-1}} e^{i a\left[z_{1} \kappa_{z}+z_{2} \operatorname{sgn}\left(\kappa_{z}\right) \sqrt{\kappa_{z}^{2}-1}\right]} \tag{51}
\end{equation*}
$$

Because $\sqrt{\kappa_{z}^{2}-1}=i \sqrt{1-\kappa_{z}^{2}}$, we have:

$$
\begin{equation*}
L_{\mathrm{sc}}=\int_{-1}^{1} d \kappa_{z} \frac{a^{2} \kappa_{z}^{2}+k_{m}^{2}}{\operatorname{sgn}\left(\kappa_{z}\right) \sqrt{1-\kappa_{z}^{2}}} e^{i a z_{1} \kappa_{z}} e^{-a z_{2} \operatorname{sgn}\left(\kappa_{z}\right) \sqrt{1-\kappa_{z}^{2}}} \tag{52}
\end{equation*}
$$

Using the fact that in this case, $a^{2}=4 k_{\lambda}^{2}-k_{m}^{2}$, we have:

$$
\begin{align*}
L_{\mathrm{sc}} & =4 k_{\lambda}^{2} \int_{-1}^{1} d \kappa_{z} \frac{\kappa_{z}^{2}}{\operatorname{sgn}\left(\kappa_{z}\right) \sqrt{1-\kappa_{z}^{2}}} e^{i a z_{1} \kappa_{z}} e^{-a z_{2} \operatorname{sgn}\left(\kappa_{z}\right) \sqrt{1-\kappa_{z}^{2}}} \\
& +k_{m}^{2} \int_{-1}^{1} d \kappa_{z} \operatorname{sgn}\left(\kappa_{z}\right) \sqrt{1-\kappa_{z}^{2}} e^{i a z_{1} \kappa_{z}} e^{-a z_{2} \operatorname{sgn}\left(\kappa_{z}\right) \sqrt{1-\kappa_{z}^{2}}}  \tag{53}\\
& =2 \pi i \times\left(4 k_{\lambda}^{2} H_{\mathrm{J} 2}\left(a z_{1}, \text { fluc }=a z 2\right)+4 k_{m}^{2} H_{\mathrm{J} 3}\left(a z_{1}, \text { fluc }=a z_{2}\right)\right) .
\end{align*}
$$

The remaining parts can be handled as before:
(1) $z_{1}+z_{2}>0$, in which (as $k_{z}$ goes from $-\infty$ to $-a$, then jumps to the interval $a$ to $\infty$ ) $\phi$ ranges from $-\infty$ to $-z_{1}$, then jumps to the interval $z_{1}$ to $\infty$.
(2) $z_{1}+z_{2}<0$, in which (again as $k_{z}$ goes from $-\infty$ to $-a$, then jumps to the interval $a$ to $\infty) \phi$ ranges from $\infty$ to $z_{1}$, then jumps up to the interval $-z_{1}$ to $-\infty$.
This integration range is summarized as follows:

$$
\begin{align*}
L_{1}+L_{2}+L_{3}+L_{4} & =\left(\operatorname{sgn}\left(z_{1}+z_{2}\right) \int_{-\infty}^{\infty}-\int_{-z_{1}}^{z_{1}}\right) i e^{-i a \phi} d \phi \\
& \times\left\{\left[\frac{a}{z_{1}+z_{2}}\right]^{2} \phi+\frac{a^{2}\left(z_{1}^{2}+z_{2}^{2}\right)}{\left(z_{2}^{2}-z_{1}^{2}\right)^{2}} \frac{\phi(\phi-\psi)}{\psi}+\left[k_{m}^{2}+\frac{a^{2} z_{2}^{2}}{z_{2}^{2}-z_{1}^{2}}\right] \frac{1}{\psi}\right\} . \tag{54}
\end{align*}
$$

The first integral $L_{1}$ can be derived the same way as in the previous case:

$$
\begin{align*}
L_{1} & =\operatorname{sgn}\left(z_{1}+z_{2}\right) \int_{-\infty}^{\infty} d \phi i e^{-i a \phi}\left(\frac{a}{z_{1}+z_{2}}\right)^{2} \phi \\
& =\frac{\operatorname{sgn}\left(z_{1}+z_{2}\right)}{\left(z_{1}+z_{2}\right)^{2}} a^{2} \int_{-\infty}^{\infty} d \phi i \phi e^{-i a \phi}  \tag{55}\\
& =\frac{\operatorname{sgn}\left(z_{1}+z_{2}\right)}{\left(z_{1}+z_{2}\right)^{2}} a^{2} \delta^{\prime}(a) \\
& =\delta^{\prime}\left(z_{1}+z_{2}\right) .
\end{align*}
$$

The second integral $L_{2}$ is:

$$
\begin{align*}
L_{2} & =-\int_{-z_{1}}^{z_{1}} d \phi i e^{-i a \phi}\left(\frac{a}{z_{1}+z_{2}}\right)^{2} \phi  \tag{56}\\
& =-\frac{2\left[a z_{1} \cos a z_{1}-\sin a z_{1}\right]}{\left(z_{1}+z_{1}\right)^{2}} .
\end{align*}
$$

The remaining parts of the integral can be computed similarly. For the third term $L_{3}$, we have:

$$
\begin{equation*}
L_{3}=\left(\operatorname{sgn}\left(z_{1}+z_{2}\right) \int_{-\infty}^{\infty}-\int_{-z_{1}}^{z_{1}}\right) i e^{-i a \phi} d \phi \frac{a^{2}\left(z_{1}^{2}+z_{2}^{2}\right)}{\left(z_{2}^{2}-z_{1}^{2}\right)^{2}} \frac{\phi(\phi-\psi)}{\psi} . \tag{57}
\end{equation*}
$$

The integral above can be simplified by making the following transforms: $z_{e}=z_{1}^{2}-z_{2}^{2}$, $z_{f}=z_{1}^{2}+z_{2}^{2}, \nu=\frac{\phi}{\sqrt{\left|z_{e}\right|}}$, and, noticing that, in this case, $\psi=\operatorname{sgn} \sqrt{\phi^{2}+\left(z_{2}^{2}-z_{1}^{2}\right)}$, we have:

$$
\begin{equation*}
L_{3}=\left(\operatorname{sgn}\left(z_{1}+z_{2}\right) \int_{-\infty}^{\infty}-\int_{-\frac{z_{1}}{\sqrt{\left|z_{e}\right|}}}^{\frac{z_{1}}{\sqrt{z^{\mid} \mid}}}\right) i e^{-i a \sqrt{\left|z_{e}\right|}} d \nu \frac{a^{2} z_{f}}{\left|z_{e}\right|} \frac{\nu\left(\nu-\operatorname{sgn} \sqrt{\nu^{2} \mp 1}\right)}{\operatorname{sgn} \sqrt{\nu^{2} \mp 1}}, \tag{58}
\end{equation*}
$$

where the minus or plus sign in the equation above depends on whether $z_{1}^{2}-z_{2}^{2}>0$ or $z_{1}^{2}-z_{2}^{2}<0$. The integral above can be easily expressed by our pre-defined functions; in the case of $z_{2}^{2}<z_{1}^{2}$, we have

$$
\begin{equation*}
L_{3}=-2 \pi \frac{a^{2} z_{f}}{\left|z_{e}\right|}\left(\operatorname{sgn}\left(z_{1}+z_{2}\right)\right) \times H_{N 2}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\operatorname{sgn}\left(z_{1}+z_{2}\right) \frac{z_{1}}{\sqrt{\left|z_{e}\right|}}\right) \tag{59}
\end{equation*}
$$

In the case of $z_{2}^{2}>z_{1}^{2}$, we have

$$
\begin{equation*}
L_{3}=-2 \pi \frac{a^{2} z_{f}}{\left|z_{e}\right|}\left(\operatorname{sgn}\left(z_{1}+z_{2}\right)\right) \times H_{K 2}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\operatorname{sgn}\left(z_{1}+z_{2}\right) \frac{z_{1}}{\sqrt{\left|z_{e}\right|}}\right) . \tag{60}
\end{equation*}
$$

For the fourth term $L_{4}$, we have:

$$
\begin{equation*}
L_{4}=\left(\operatorname{sgn}\left(z_{1}+z_{2}\right) \int_{-\infty}^{\infty}-\int_{-z_{1}}^{z_{1}}\right) i e^{-i a \phi} d \phi\left(k_{m}^{2}+\frac{a^{2} z_{2}^{2}}{z_{2}^{2}-z_{1}^{2}}\right) \frac{1}{\psi} \tag{61}
\end{equation*}
$$

Changing variables as we did to solve for $L_{3}$, we can similarly calculate $L_{4}$.

$$
\begin{equation*}
L_{4}=\left(\operatorname{sgn}\left(z_{1}+z_{2}\right) \int_{-\infty}^{\infty}-\int_{-\frac{z_{1}}{\sqrt{|z e|}}}^{\frac{z_{1}}{\sqrt{|z|}}}\right) i e^{-i a \sqrt{\left|z_{e}\right| \nu}} d \nu\left(k_{m}^{2}+\frac{a^{2} z_{2}^{2}}{z_{2}^{2}-z_{1}^{2}}\right) \frac{1}{\sqrt{\nu^{2} \mp 1}} \tag{62}
\end{equation*}
$$

where the minus or plus sign in the equation above again depends on whether $z_{1}^{2}-z_{2}^{2}>0$ or $z_{1}^{2}-z_{2}^{2}<0$. The integral above is exactly the integral we defined in Appendix B, so In the case of $z_{2}^{2}<z_{1}^{2}$, we have therefore

$$
\begin{equation*}
L_{4}=2 \pi \frac{4 k_{\lambda}^{2} z_{2}^{2}-k_{m}^{2} z_{1}^{2}}{z_{e}}\left(\operatorname{sgn}\left(z_{1}+z_{2}\right)\right) \times H_{N}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\operatorname{sgn}\left(z_{1}+z_{2}\right) \frac{z_{1}}{\sqrt{\left|z_{e}\right|}}\right) \tag{63}
\end{equation*}
$$

and in the case of $z_{2}^{2}>z_{1}^{2}$, we have

$$
\begin{equation*}
L_{4}=2 \pi \frac{4 k_{\lambda}^{2} z_{2}^{2}-k_{m}^{2} z_{1}^{2}}{z_{e}}\left(\operatorname{sgn}\left(z_{1}+z_{2}\right)\right) \times H_{K}\left(a \sqrt{\left|z_{e}\right|}, \text { trunc }=\operatorname{sgn}\left(z_{1}+z_{2}\right) \frac{z_{1}}{\sqrt{\left|z_{e}\right|}}\right) \tag{64}
\end{equation*}
$$

## Appendix B

In this appendix we itemize some Fourier integrals used in the body of the paper. We have:

$$
\begin{gather*}
H_{J}(x, \text { fluc }=y)=\frac{1}{2 \pi} \int_{-1}^{1} d \omega \frac{1}{i \operatorname{sgn}(\omega) \sqrt{1-\omega^{2}}} e^{i \omega x} e^{-y \operatorname{sgn}(\omega) \sqrt{1-\omega^{2}}}  \tag{65}\\
H_{J 2}(x, \text { fluc }=y)=\frac{1}{2 \pi} \int_{-1}^{1} d \omega \frac{\omega^{2}}{i \operatorname{sgn}(\omega) \sqrt{1-\omega^{2}}} e^{i \omega x} e^{-y \operatorname{sgn}(\omega) \sqrt{1-\omega^{2}}}  \tag{66}\\
H_{J 3}(x, \text { fluc }=y)=\frac{1}{2 \pi} \int_{-1}^{1} d \omega \frac{\sqrt{1-\omega^{2}}}{i \operatorname{sgn}(\omega)} e^{i \omega x} e^{-y \operatorname{sgn}(\omega) \sqrt{1-\omega^{2}}}  \tag{67}\\
=H_{J}(x, \text { fluc }=y)-H_{J 2}(x, \text { fluc }=y) \\
H_{K}\left(x, \operatorname{trunc}=d_{1}\right)=\frac{1}{2 \pi}\left(\int_{-\infty}^{\infty}-\operatorname{sgn}\left(d_{1}\right) \int_{-\left|d_{1}\right|}^{\left|d_{1}\right|}\right) \frac{e^{i \omega x}}{i \operatorname{sgn}(\omega) \sqrt{\omega^{2}+1}} d \omega  \tag{68}\\
H_{K 2}\left(x, \text { trunc }=d_{1}\right)=\frac{1}{2 \pi}\left(\int_{-\infty}^{\infty}-\operatorname{sgn}\left(d_{1}\right) \int_{-\left|d_{1}\right|}^{\left|d_{1}\right|}\right) \frac{e^{i \omega x}}{i \operatorname{sgn}(\omega)}\left(\frac{\omega^{2}}{\sqrt{\omega^{2}+1}}-|\omega|\right) d \omega \tag{69}
\end{gather*}
$$

$$
\begin{gather*}
H_{N}\left(x, \text { trunc }=d_{1}\right)=\frac{1}{2 \pi}\left(\int_{|\omega|>1}-\operatorname{sgn}\left(d_{1}\right) \int_{\left|d_{2}\right|>|\omega|>1}\right) \frac{e^{i \omega x}}{i \operatorname{sgn}(\omega) \sqrt{\omega^{2}-1}} d \omega  \tag{70}\\
H_{N 2}\left(x, \text { trunc }=d_{1}\right)=\frac{1}{2 \pi}\left(\int_{|\omega|>1}-\operatorname{sgn}\left(d_{1}\right) \int_{\left|d_{2}\right|>|\omega|>1}\right) \frac{e^{i \omega x}}{i \operatorname{sgn}(\omega)}\left(\frac{\omega^{2}}{\sqrt{\omega^{2}-1}}-|\omega|\right) d \omega \tag{71}
\end{gather*}
$$

## Appendix C

In this appendix we demonstrate the reduction of the general derivation of $\alpha_{2}(x, z)$ to the previously developed and tested 1D form $\alpha_{2}(z)$.
The data may be expressed as

$$
\begin{align*}
& D\left(k_{g}=0.5 k_{m}, z_{g}, k_{s}=-0.5 k_{m}, z_{s}, \omega\right) \\
& =\int_{-\infty}^{\infty} d x_{g} e^{-i k_{g} x_{g}} \int_{-\infty}^{\infty} d x_{s} e^{i k_{s} x_{s}} D\left(x_{g}, z_{g}, x_{s}, z_{s}, \omega\right) \\
& =\int_{-\infty}^{\infty} d x_{g} e^{-i\left(k_{m} / 2\right) x_{g}} \int_{-\infty}^{\infty} d x_{s} e^{-i\left(k_{m} / 2\right) x_{s}} D\left(x_{g}, z_{g}, x_{s}, z_{s}, \omega\right)  \tag{72}\\
& =\int_{-\infty}^{\infty} d x_{g} e^{-i\left(k_{m} / 2\right) x_{g}} \int_{-\infty}^{\infty} d x_{s} e^{-i\left(k_{m} / 2\right) x_{s}} D\left(z_{g}, z_{s}, x_{g}-x_{s} \omega\right)
\end{align*}
$$

in which we make use of the fact that in 1D the data depends only on the offset coordinate and not the midpoint coordinate (that is, all shot-record like experiments are identical). The Fourier transform with respect to midpoint is therefore a delta-function:

$$
\begin{equation*}
D\left(k_{g}=0.5 k_{m}, z_{g}, k_{s}=-0.5 k_{m}, z_{s}, \omega\right)=2 \pi \delta\left(k_{m}\right) \int_{-\infty}^{\infty} d x_{h} D\left(z_{g}, z_{s}, x_{h}, \omega\right) \tag{73}
\end{equation*}
$$

Recognizing $\int_{-\infty}^{\infty} D\left(z_{g}, z_{s}, x_{h}, \omega\right) d x_{h}$ as $D\left(z_{g}, z_{s}, \omega\right)$, the data used in Shaw et al. (2002) and Zhang and Weglein (2002), the latter at normal incidence and for invariant density. The delta function itself sifts out the component $k_{m}=0$ :

$$
\begin{align*}
\widetilde{\alpha}_{1}\left(k_{m}, z\right) & =-\frac{c_{0}^{2}}{2 \pi \rho_{r}} \int_{-\infty}^{\infty} d k_{z} \frac{k_{z}^{2}}{\omega^{2}} e^{-i k_{z}\left[z-0.5\left(z_{g}+z_{s}\right)\right]} D\left(\frac{k_{m}}{2}, z_{g},-\frac{k_{m}}{2}, z_{s}, \omega\right) \\
& =-\frac{c_{0}^{2}}{2 \pi \rho_{r}} \int_{-\infty}^{\infty} d k_{z} \frac{k_{z}^{2}}{\omega^{2}} e^{-i k_{z}\left[z-0.5\left(z_{g}+z_{s}\right)\right]} 2 \pi \delta\left(k_{m}\right) D\left(z_{g}, z_{s}, \omega\right) \\
& =-\frac{c_{0}^{2} \delta\left(k_{m}\right)}{\rho_{r}} \int_{-\infty}^{\infty} d k_{z} \frac{k_{z}^{2}}{\omega^{2}} e^{-i k_{z}\left[z-0.5\left(z_{g}+z_{s}\right)\right]} D\left(z_{g}, z_{s}, \omega\right)  \tag{74}\\
& =-\frac{c_{0}^{2} \delta\left(k_{m}\right)}{\rho_{r}} \int_{-\infty}^{\infty} d k_{z} \frac{4}{c_{0}^{2}} e^{-i k_{z}\left[z-0.5\left(z_{g}+z_{s}\right)\right]} D\left(z_{g}, z_{s}, \omega\right) \\
& =-\frac{4 \delta\left(k_{m}\right)}{\rho_{r}} \int_{-\infty}^{\infty} d k_{z} e^{-i k_{z}\left[z-0.5\left(z_{g}+z_{s}\right)\right]} D\left(z_{g}, z_{s}, \omega\right) \\
& =2 \pi \delta\left(k_{m}\right) \alpha_{1}(z) .
\end{align*}
$$

Here $\alpha_{1}(z)$ is recognizable as the 1D linear perturbation. The second term is then computed using equation (74):

$$
\begin{align*}
\widetilde{\alpha}_{2}\left(k_{m}, z\right) & =\frac{1}{16 \pi^{2}} \int_{-\infty}^{\infty} d k_{\lambda} \int_{0}^{\infty} d z^{\prime} \widetilde{\alpha}_{1}\left(0.5 k_{m}-k_{\lambda}, z^{\prime}\right) \\
& \times\left\{\int_{0}^{\infty} d z^{\prime \prime} \widetilde{\alpha}_{1}\left(k_{\lambda}+0.5 k_{m}, z^{\prime \prime}\right)\right\} \gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right) \\
& =\frac{1}{16 \pi^{2}} \int_{-\infty}^{\infty} d k_{\lambda} \int_{0}^{\infty} d z^{\prime} 2 \pi \delta\left(0.5 k_{m}-k_{\lambda}\right) \widetilde{\alpha}_{1}\left(z^{\prime}\right) \\
& \times \int_{0}^{\infty} d z^{\prime \prime} 2 \pi \delta\left(k_{\lambda}+0.5 k_{m}\right) \widetilde{\alpha}_{1}\left(z^{\prime \prime}\right) \gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right) \\
& =\frac{1}{4} \int_{-\infty}^{\infty} d k_{\lambda} \int_{0}^{\infty} d z^{\prime} \delta\left(0.5 k_{m}-k_{\lambda}\right) \widetilde{\alpha}_{1}\left(z^{\prime}\right)  \tag{75}\\
& \times \int_{0}^{\infty} d z^{\prime \prime} \delta\left(k_{\lambda}+0.5 k_{m}\right) \widetilde{\alpha}_{1}\left(z^{\prime \prime}\right) \gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right) \\
& =\frac{1}{4} \int_{0}^{\infty} d z^{\prime} \widetilde{\alpha}_{1}\left(z^{\prime}\right) \int_{0}^{\infty} d z^{\prime \prime} \widetilde{\alpha}_{1}\left(z^{\prime \prime}\right) \\
& \times \int_{-\infty}^{\infty} d k_{\lambda} \delta\left(0.5 k_{m}-k_{\lambda}\right) \delta\left(k_{\lambda}+0.5 k_{m}\right) \gamma\left(z, z^{\prime}, z^{\prime \prime}, k_{\lambda}\right) .
\end{align*}
$$

We simplify by recognizing that since

$$
\begin{equation*}
\left(k_{z}^{2}+k_{m}^{2}\right) \delta\left(k_{m}\right)=k_{z}^{2} \delta\left(k_{m}\right), \tag{76}
\end{equation*}
$$

we have

$$
\begin{align*}
\gamma\left(z, z^{\prime}, z^{\prime \prime}, 0.5 k_{m}\right) & =i \int_{-\infty}^{\infty} d k_{z} \frac{k_{z}^{2}}{k_{z}} e^{i\left(z_{1}+z_{2}\right) k_{z}} \\
& =2 \pi \delta^{\prime}\left(z_{1}+z_{2}\right)  \tag{77}\\
& =2 \pi \delta^{\prime}\left(0.5\left(z^{\prime}+z^{\prime \prime}\right)-z+0.5\left|z^{\prime}-z^{\prime \prime}\right|\right) .
\end{align*}
$$

This simplified $\gamma$ may then be substituted into equation (75):

$$
\begin{align*}
\widetilde{\alpha}_{2}\left(k_{m}, z\right) & =\frac{\delta\left(k_{m}\right)}{4} \int_{0}^{\infty} d z^{\prime} \alpha_{1}\left(z^{\prime}\right) \int_{0}^{\infty} d z^{\prime \prime} \alpha_{1}\left(z^{\prime \prime}\right) \\
& \times 2 \pi \delta^{\prime}\left(0.5\left(z^{\prime}+z^{\prime \prime}\right)-z+0.5\left|z^{\prime}-z^{\prime \prime}\right|\right) \\
& =2 \pi \frac{\delta\left(k_{m}\right)}{4} \int_{0}^{\infty} d z^{\prime} \alpha_{1}\left(z^{\prime}\right) \int_{0}^{\infty} d z^{\prime \prime} \alpha_{1}\left(z^{\prime \prime}\right) \\
& \times 2 \pi \delta^{\prime}\left(0.5\left(z^{\prime}+z^{\prime \prime}\right)-z+0.5\left|z^{\prime}-z^{\prime \prime}\right|\right) \\
& =2 \pi \delta\left(k_{m}\right) \frac{1}{4} \int_{0}^{\infty} d z^{\prime} \alpha_{1}\left(z^{\prime}\right)  \tag{78}\\
& \times\left\{\int_{0}^{z^{\prime}} d z^{\prime \prime} \alpha_{1}\left(z^{\prime \prime}\right) \delta^{\prime}\left(z^{\prime}-z\right)+\int_{z^{\prime}}^{\infty} d z^{\prime \prime} \alpha_{1}\left(z^{\prime \prime}\right) \delta^{\prime}\left(z^{\prime \prime}-z\right)\right\} \\
& =2 \pi \delta\left(k_{m}\right)\left(-\frac{1}{2}\right)\left[\alpha_{1}^{2}(z)+\alpha_{1}^{\prime}(z) \int_{0}^{z} d z^{\prime} \alpha_{1}\left(z^{\prime}\right)\right] .
\end{align*}
$$

Defining $\widetilde{\alpha}_{2}\left(k_{m}, z\right)=2 \pi \delta\left(k_{m}\right) \alpha_{2}(z)$, as in the linear term, we have

$$
\begin{equation*}
\alpha_{2}(z)=-\frac{1}{2}\left[\alpha_{1}^{2}(z)+\alpha_{1}^{\prime}(z) \int_{0}^{z} d z^{\prime} \alpha_{1}\left(z^{\prime}\right)\right] . \tag{79}
\end{equation*}
$$

This is the expression for $\alpha_{2}(z)$ in the 1D case, as desired.

# Imaging with $\tau=0$ versus $t=0$ : towards including headwaves into imaging and internal multiple attenuation theory 

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#### Abstract

The internal multiples attenuation algorithm derived from inverse scattering series techniques can be interpreted as a sequence of un-collapsed migrations with the reference medium velocity and restricted to lower-higher-lower pseudo-depths (Weglein et al. 1997, Weglein et al. 2003). The arrivals in the data are regarded as sub-events in the creation of a multiple. These sub-events are imaged through these migrations with $t=0$ imaging condition. In this paper we discuss both the $t=0$ and the $\tau=0$ imaging condition and show that the latter is more general in its ability to image, in addition, events that exhibit a horizontal propagation part in their path (e.g. headwaves along a horizontal interface). This research is part of our strategy to include more of the returning signal, information bearing or not, in our analysis and processing.


## 1 Introduction

The inverse scattering series is a multi-dimensional inversion procedure that directly determines physical properties using only recorded data and a reference medium. This inversion process can be thought of as performing the following four tasks: (1) free surface multiple removal, (2) internal multiple removal, (3) location of reflectors in space and (4) identification of medium property changes across reflectors. These tasks were associated with subseries of the full series, subseries which, if identified, would perform their job as if no other task existed in the series. Two immediate advantages of this separation of tasks are the favorable convergence properties of the subseries and the ability to judge the effectiveness of each step before proceeding on to the next. Since the entire process requires only data and reference medium information, it is reasonable to assume that intermediate steps that are associated with achieving that objective would also be attainable with only the reference medium and data.

Subseries that exhibit this property have been identified for all four tasks (Weglein et al. 2003 and references therein). Algorithms resulting from the subseries for the task of free surface and internal multiple attenuation have been successfully applied to field data (Weglein et al. 2003). The internal multiples attenuation algorithm derived from inverse scattering series techniques can be interpreted as a sequence of un-collapsed migrations with the reference
medium velocity and restricted to lower-higher-lower pseudo-depths (Weglein et al. 1997, Weglein et al. 2003). The arrivals in the data are regarded as sub-events in the creation of a multiple. The sub-events are imaged through these migrations with $t=0$ imaging condition. In this paper we discuss the $t=0$ versus the $\tau=0$ imaging condition and show that the latter is more general in its ability to image, in addition, events that exhibit a horizontal propagation part in their path, e.g. headwaves along a horizontal interface and reflections from dipping reflectors. The purpose of this research is to expand our understanding of the events and sub-events in the returning signal and it is part of our strategy to include more of this signal, information bearing or not, in our analysis and processing.

The idea of using $\tau=0$ as an imaging condition has been used by Clayton and McMechan (1981) to image refraction data to produce velocity-depth profiles from recorded data. Their method involves a slant-stack of the data to produce a wavefield in the $p-\tau$ domain, where $p$ is the horizontal slowness, and a downward continuation and imaging with $\tau=0$.

The plan for this paper is as follows. In Section 2 we give a brief description of the internal multiples attenuation algorithm derived from the inverse scattering series techniques. In Section 3 we present headwaves as prime events or sub-events of composite events using both ray and wave front diagrams; in Section 4, we briefly discuss the pre-stack constant velocity phase-shift migration. Section 5 presents a comparison between the travel time $t$ and the vertical time $\tau$. In Section 6 we present an analytic example of imaging headwaves from horizontal interfaces using $\tau=0$ imaging condition. Some conclusions are drawn in Section 7. Throughout the paper, "horizontal" refers to the direction parallel to the measurement surface in a 3D seismic experiment.

## 2 Internal multiples attenuation algorithm

The second term in the inverse scattering subseries for internal multiple attenuation is (see e.g. Weglein et al 2003)

$$
\begin{align*}
b_{3}\left(k_{g}, k_{s}, q_{g}+q_{s}\right) & =\frac{1}{(2 \pi)^{2}} \iint d k_{1} e^{-i q_{1}\left(\epsilon_{g}-\epsilon_{s}\right)} d k_{2} e^{i q_{2}\left(\epsilon_{g}-\epsilon_{s}\right)} \\
& \times \int_{-\infty}^{\infty} d z_{1} e^{i\left(q_{g}+q_{1}\right) z_{1}} b_{1}\left(k_{g}, k_{1}, z_{1}\right) \int_{-\infty}^{z_{1}} d z_{2} e^{i\left(-q_{1}-q_{2}\right) z_{2}} b_{1}\left(k_{1}, k_{2}, z_{2}\right) \\
& \times \int_{z_{2}}^{\infty} d z_{3} e^{i\left(q_{2}+q_{s}\right) z_{3}} b_{1}\left(k_{2}, k_{s}, z_{3}\right) \tag{1}
\end{align*}
$$

where $z_{1}>z_{2}$ and $z_{2}>z_{3}$ and $b_{1}$ is defined in terms of the original pre-stack data with free surface multiples eliminated, $D^{\prime}$, to be

$$
\begin{equation*}
D^{\prime}\left(k_{g}, k_{s}, \omega\right)=\left(-2 i q_{s}\right)^{-1} B(\omega) b_{1}\left(k_{g}, k_{s}, q_{g}+q_{s}\right) \tag{2}
\end{equation*}
$$

with $B(\omega)$ being the source signature.
The terms $b_{1}\left(k_{g}, k_{s}, z\right)$ in formula (1) can be thought of as being obtained through the following procedure. Start with the effective data in the $f-k$ domain, $b_{1}\left(k_{g}, k_{s}, \omega\right)$, and downward continue the source and receiver by applying a phase-shift $e^{i k_{z} z} b_{1}\left(k_{g}, k_{s}, \omega\right)$. Subsequent integration over $k_{z}$ to obtain $b_{1}\left(k_{g}, k_{s}, z\right)$ is a simple Jacobian away from integration over $\omega$ ( $t=0$ imaging condition). The algorithm can hence be interpreted as a sequence of un-collapsed migrations restricted to lower-higher-lower pseudo-depths. Notice that any arrival in the data is regarded as a sub-event by the algorithm and imaged through the process above. Three prime events in the data will create, and hence attenuate, a first order multiple; subsequent composite arrivals will create and attenuate higher order multiples.

The presence of an implicit $t=0$ imaging condition in the algorithm motivates its comparison against imaging with $\tau=0$, where $\tau$ is the vertical time. We found the latter to be more general in its ability to collapse any horizontal propagations and hence to image properly, in addition, events which contain horizontal parts in their propagation paths, e.g. headwaves along horizontal interfaces. To be able to image a larger class of events (or sub-events) using a different imaging condition means to be able to attenuate a larger class of internal multiples and reduce the number of artifacts in the de-multipled data.

Although the rest of the paper mainly discusses headwaves due to horizontal interfaces we mention that the results apply to all prime events or sub-events of composite events which contain horizontally propagating parts.

## 3 Headwaves as prime events or sub-events of composite events

The propagation path of the headwaves (see Figure 1), also known in the literature as conical or lateral waves, was first recognized by Mohorovicic during his studies of the arrival time of certain waves from an earthquake in 1909. The headwave has a linear relationship between arrival time and horizontal range and, at sufficient offset, it is the first arriving wave (attribute which accounts for its name). The one who originated the theory of headwaves as recognizably distinct arrivals was Jeffreys (1926) although the source/medium geometries needed for such a wave to develop have been known since 1904 and referred under the general term "Lamb's Problem". An excellent brief history of the headwaves can be found in Cerveny and Ravindra (1971).

A simple picture of the headwave can be given using the Huygens Principle which was first applied to these problems by Merten (1927) and is described in many books on seismology or wave propagation. The simplest case, in which the physical conditions for headwaves to occur are satisfied, is that of two semi-infinite homogeneous liquid media. We assume that the point source is located at $s$ at a distance $z_{1}$ from a horizontal interface separating two


Figure 1: A ray picture of the headwave.
media such that the velocity of propagation in the second medium, $c_{1}$, is higher than the velocity of propagation in the first medium, $c_{0}$ (see Figure 2).

The cylindrical coordinates of an arbitrary point are $(r, z, \varphi)$ where $r$ is the distance from the vertical axis in a horizontal plane. We will consider only the two dimensional section $(r, z)$ with $\varphi=$ const. Assume that the source starts to emit waves at time $t=0$. For $t<z_{1} / c_{0}$, i.e. before the wave reaches the interface, only the incident wave exists (see Figure 2a). In this picture, the point $C$ indicates the critical incidence. The wavefront is a sphere with the center at $s$ and radius $R=\sqrt{r^{2}+\left(z-z_{1}\right)^{2}}$ proportional to $t$ (i.e. $t=R / c_{0}$ ). For $t=z_{1} / c_{0}$, the wavefront of the incident wave reaches the interface and it is tangent to it. As $t$ increases further, reflected and refracted waves appear, as each point on the interface hit by the incidence wave becomes a source of disturbance according to Huygens principle. The wave fronts of these waves for $z_{1} / c_{0}<t<z_{1} /\left(c_{0} \cos i_{c}\right)$ are shown in Figure 2 b . The wave fronts of the incident, reflected and transmitted waves are connected at the point $P$ on the interface which moves along the interface with increasing time. The speed of the point $P$ along the interface can be calculated to be

$$
\begin{equation*}
c_{P}=\frac{c_{0}}{\sin i_{P}} \tag{3}
\end{equation*}
$$

where $i_{P}$ is the angle between the ray incident at the point $P$ and the vertical. The angle $i_{P}$ increases with horizontal distance $r$, and so does $\sin i_{P}$ which means that the velocity of


Figure 2: Wavefront diagrams showing the development of the headwave.
the point $P$ along the interface, $c_{P}$, decreases. While $c_{P}>c_{1}$ the situation remains the same with all three wavefronts connected at the point $P$. However, for $t=z_{1} /\left(c_{0} \cos i_{c}\right)$, where $i_{c}=\sin ^{-1}\left(c_{0} / c_{1}\right)$, the point $P$ reaches the point $C$ where we have

$$
\begin{equation*}
c_{P}=\frac{c_{0}}{\sin i_{c}}=c_{1} . \tag{4}
\end{equation*}
$$

For larger $t$, we have $c_{P}<c_{1}$ and the transmitted wave, propagating from the point $C$ in the second medium will be more advanced than the incident and reflected waves (see Figure ??c). Points on the interface which are reached by the refracted wave first, i.e. all the points with $r$ coordinate bigger than the $r$ coordinate of the point $C$, will become centers of disturbances propagating back into the first medium with velocity $c_{0}$. These disturbances form the headwave, the envelop of these propagations being its wave front. As the velocities $c_{0}$ and $c_{1}$ are constant, the wavefront of the headwave is a straight line (in three dimensions it is the frustum of a cone).

An important feature that emerges from the wavefront diagram representation outlined above is that the headwaves are due to the curvature of the wavefront and hence it would be impossible to create headwaves if the wave impinging on the interface would be a planewave. This, and the fact that the headwave itself is a plane-wave, leads to the conclusion that a headwave cannot create a multiple of itself and hence the seismic event pictured in Figure 3 is not a real one.


Figure 3: An impossible event in which headwaves are sub-events.
However, headwaves can occur as sub-events in a composite event if for example the other sub-events are regular reflections; we call these kind of events, i.e. containing a headwave as a sub-event, refracted multiples. The refracted multiples can be divided into free-surface and internal refracted multiples depending whether any downward reflection takes place at the free-surface, or they all take place inside the actual medium (Figure 4 shows examples of first order internal refracted multiples).

The free surface refracted multiples are presently removed by free-surface de-multiple algorithms (Dragoset (2003)). In this paper we show that, while the $t=0$ imaging condition is not physically appropriate to handle events containing horizontally propagating parts (e.g. headwaves from horizontal reflectors), the $\tau=0$ imaging condition has the ability to image them at the correct depth. An analytic example showing this is given in Section 6.


Figure 4: First order internal refracted multiples.

## 4 Pre-stack constant velocity phase-shift migration

In this section we give a brief mathematical account of the pre-stack constant velocity phaseshift migration following Stolt and Benson (1986).

The data $D\left(x_{g}, y_{g} \mid x_{s}, y_{s} ; t\right)$ recorded on the measurement surface is the expression of an up-going wavefield $P$ calculated at $z=0$, i.e.

$$
\begin{equation*}
D\left(x_{g}, y_{g} \mid x_{s}, y_{s} ; t\right)=P\left(x_{g}, y_{g}, 0 \mid x_{s}, y_{s}, 0 ; t\right) \tag{5}
\end{equation*}
$$

In this equation $t$ represents the travel-time of the wave, i.e. the time it takes for the signal to travel from the source to the reflector and back to the receiver. The amplitude of the returning signal is a function which depends on the reflectivity $M(x, y, z)$ where $(x, y, z)$ denotes a point on the reflector. Migration is the operation of mapping the data $D$ onto the reflectivity $M$

$$
\begin{equation*}
D\left(x_{g}, y_{g} \mid x_{s}, y_{s} ; t\right) \rightarrow M(x, y, z) \tag{6}
\end{equation*}
$$

This mapping is achieved in two steps with the use of the wavefield $P: 1$. The first step, the downward continuation, is to derive the wavefield at any depth $P\left(x_{g}, y_{g}, z \mid x_{s}, y_{s}, z ; t\right)$ from the wavefield at the surface $P\left(x_{g}, y_{g}, 0 \mid x_{s}, y_{s}, 0 ; t\right)$. The downward continuation is possible because $P$ is an up-going solution to the scalar wave equation, hence providing all the necessary information for the extrapolation. 2. The second step, the imaging, is to restrict $P\left(x_{g}, y_{g}, z \mid x_{s}, y_{s}, z ; t\right)$ by applying the so called imaging condition $t=0$ and hence obtaining a quantity which is a function of reflectivity, i.e. $P\left(x_{g}, y_{g}, z \mid x_{s}, y_{s}, z ; 0\right)$. The goal of this second step is to pinpoint the exact moment when the wave "turns", i.e. transforms from a down-going into an up-going wave due to the interaction with the reflector. The two steps, and hence the entire migration concept, can be expressed as

$$
\begin{equation*}
M(x, y, z)=F[P(x, y, z \mid x, y, z ; 0)] \tag{7}
\end{equation*}
$$

i.e. the reflectivity $M$ is a function $F$ of the extrapolated data $P$ at time $t=0$. The simplest choice of $F$ is the unit operator although this is not the preferred choice (see Stolt and Benson Ch 3).
When the velocity is constant, the wavefield at the measurement surface (the data) can be decomposed into plane-wave components and the extrapolation at any depth $z$ can be
obtained simply by applying a phase-shift $e^{i k_{z} z}$, on both source and receiver, to each component, where $k_{z}$ is the vertical wave-number of the plane-wave component being downward continued. To express this notion in mathematics, we first decompose $P$ to planewaves using a Fourier Transform written as

$$
\begin{align*}
& P\left(k_{g x}, k_{g y}, 0 \mid k_{s x}, k_{s y}, 0 ; \omega\right)=\int d x_{g} \int d y_{g} \int d x_{s} \int d y_{s} \int d t \\
& e^{i\left(\omega t-k_{g x} x_{g}-k_{g y} y_{g}+k_{s x} x_{s}+k_{s y} y_{s}\right)} P\left(x_{g}, y_{g}, 0 \mid x_{s}, y_{s}, 0 ; t\right) \tag{8}
\end{align*}
$$

where $k_{g x}^{2}+k_{g y}^{2}+k_{g z}^{2}=\omega^{2} / c_{0}^{2}$ and $k_{s x}^{2}+k_{s y}^{2}+k_{s z}^{2}=\omega^{2} / c_{0}^{2}$ with $c_{0}$ the wave speed in the reference medium. This dispersion relation fixes $k_{g z}$ and $k_{s z}$ once the other parameters are chosen so that

$$
\begin{equation*}
k_{g z}=-\operatorname{sgn}(\omega) \sqrt{\frac{\omega^{2}}{c_{0}^{2}}-k_{g x}^{2}-k_{g y}^{2}} \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
k_{s z}=-\operatorname{sgn}(\omega) \sqrt{\frac{\omega^{2}}{c_{0}^{2}}-k_{s x}^{2}-k_{s y}^{2}} \tag{10}
\end{equation*}
$$

Notice that both wavenumbers have the same sign even though the extrapolation of source coordinates carries different sign than the extrapolation of receiver coordinates. This happens however because, for reflection data, we have a down-going wave on the source side and an up-going wave on the receiver side which changes the signs again hence canceling the previous effect.

The downward continuation of both source and receiver at a common depth $z$ takes the form

$$
\begin{equation*}
P\left(k_{g x}, k_{g y}, z \mid k_{s x}, k_{s y}, z ; \omega\right)=P\left(k_{g x}, k_{g y}, 0 \mid k_{s x}, k_{s y}, 0 ; \omega\right) e^{i\left(k_{g z}+k_{s z}\right) z} \tag{11}
\end{equation*}
$$

with $k_{g z}$ and $k_{s z}$ given by equations (9) and (10). To obtain $P$ in the space domain we have to take the inverse Fourier Transform

$$
\begin{align*}
& P\left(x_{g}, y_{g}, z \mid x_{s}, y_{s}, z ; t\right)=\frac{1}{(2 \pi)^{5}} \int d k_{g x} \int d k_{g y} \int d k_{s x} \int d k_{s y} \int d \omega  \tag{12}\\
& e^{i\left(-\omega t+k_{g x} x_{g}+k_{g y} y_{g}-k_{s x} x_{s}-k_{s y} y_{s}\right)} P\left(k_{g x}, k_{g y}, z \mid k_{s x}, k_{s y}, z ; \omega\right),
\end{align*}
$$

and by setting $t=0$ in the above equation we obtain

$$
\begin{align*}
P\left(x_{g}, y_{g}, z \mid x_{s}, y_{s}, z ; 0\right) & =\frac{1}{(2 \pi)^{5}} \int d k_{g x} \int d k_{g y} \int d k_{s x} \int d k_{s y} \int d \omega  \tag{13}\\
& e^{i\left(k_{g x} x_{g}+k_{g y} y_{g}-k_{s x} x_{s}-k_{s y} y_{s}\right)} P\left(k_{g x}, k_{g y}, z \mid k_{s x}, k_{s y}, z ; \omega\right) .
\end{align*}
$$

The implementation of equation (13) is called phase-shift migration.

## 5 A comparison between travel time $t$ and intercept time $\tau$

The imaging concept in the migration procedure described in Section 4, assumes that the turning point of the wavefield is a point in space and hence, by restricting the time $t$ to zero, we would obtain the location where an up-going wave would co-exist with the first arrival of a down-going wave, hence the position in space of the reflector. While this is true for regular reflections, it is not true for events which contain horizontal parts in their propagation paths. As Figure 1 shows, the turning point of a headwave is not a point in space but an entire linear horizontal propagation. In consequence, an imaging condition $t=0$ on the data which contains these kind of events would interpret them as regular reflections and hence it would "create" reflectors at wrong depths to accommodate them. In this section we describe the connection and differences between the travel time $t$ and the intercept time $\tau$ and show that the imaging condition $\tau=0$ is a generalization of $t=0$ which would image the headwaves from horizontal interfaces at the correct depth.


Figure 5: The definition of the vertical time $\tau$.
By definition, the intercept or vertical time $\tau$ of an event arriving at offset $r_{0}$ in travel-time $t_{0}$ is the vertical component of the travel-time $t_{0}$ or, in other words, $t_{0}$ projected to zero offset along a line of slope $p$ through the point $\left(r_{0}, t_{0}\right)$. (see Figure 5). The slope $p$ of the tangent to the curve representing a reflection in the shot record pictured can be calculated as

$$
\begin{equation*}
p=\left(\frac{d t}{d r}\right)_{\left(r_{0}, t_{0}\right)} \tag{14}
\end{equation*}
$$

and hence it represent the horizontal slowness associated with that particular arrival. The equation of the tangent line gives a relationship between the travel-time $t$ and the vertical time $\tau$ of a particular arrival

$$
\begin{equation*}
t=\tau+p r . \tag{15}
\end{equation*}
$$

This formula represents a decomposition of the total time into a horizontal time, pr, and a vertical time $\tau$. To better understand this decomposition, consider a plane-wave component of constant horizontal slowness $p$ of a wavefield produced by a 3D point source and moving through a medium with constant velocity $c_{0}$. The points of intersections of this planewave
with the horizontal interface and the vertical line, move along them with constant respective speeds $c_{H}$ and $c_{V}$ such that

$$
\begin{equation*}
\sin i=\frac{c_{0}}{c_{H}} \tag{16}
\end{equation*}
$$

and

$$
\begin{equation*}
\cos i=\frac{c_{0}}{c_{V}} \tag{17}
\end{equation*}
$$

(see Figure 6a). The horizontal and the vertical slowness, $p$ and $q$, are defined as

$$
\begin{equation*}
p=\frac{1}{c_{H}}=\frac{\sin i}{c_{0}} \tag{18}
\end{equation*}
$$

and

$$
\begin{equation*}
q=\frac{1}{c_{V}}=\frac{\cos i}{c_{0}} \tag{19}
\end{equation*}
$$

and they are related through

$$
\begin{equation*}
p^{2}+q^{2}=\frac{1}{c_{0}^{2}} \tag{20}
\end{equation*}
$$



Figure 6: The relationship between $t$ and $\tau$.
Notice (from Figure 6b) that for the plane-wave to go from the source $s$ to the incidence point $I$ in time $t$ is equivalent for the projection point onto the vertical to go from $s$ to $A$ along the vertical with the speed $c_{V}$ and for the projection point onto the horizontal to go from $A$ to $I$ along the interface with the speed $c_{H}$. In this way the total time $t$ is decomposed into two parts, a vertical and a horizontal time corresponding to the horizontal and vertical motion of the projection points as follows

$$
\begin{equation*}
t=\frac{z_{1}}{c_{V}}+\frac{r}{c_{H}}=q z_{1}+p r=\tau+p r . \tag{21}
\end{equation*}
$$

From the previous equation (21), the condition $t=0$ always implies $\tau=0$. The converse is not true. For a regular reflection $\tau=0$ does imply that $t=0$. To show this notice that $\tau=0$ implies that no vertical propagation takes place, hence $z_{1}=0$. However for this type
of event there is a relationship between the horizontal and the vertical coordinates, $r$ and $z$, namely

$$
\begin{equation*}
z \tan i=r \tag{22}
\end{equation*}
$$

and so $z_{1}=0$ implies $r=0$ and they both imply $t=0$ (from equation (21)).
The same statement (and argument to prove it) applies to other events for which there is a similar relationship between the horizontal and the vertical coordinates (for example for a turning wave). However this is not true for all seismic events. For events that contain horizontal parts in their propagation paths, for example a headwave from a horizontal interface, there is no relationship between $r$ and $z$; in fact, for the part where the ray travels horizontally, we have $z=0$ (no vertical propagation) while $r \neq 0$. In this case it is obvious that $\tau=0$ does not imply $t=0$.


Figure 7: Events in $t$ and $\tau$ : the left column shows a reflection and a headwave in traveltime $t$; the right column is showing the same events in the vertical time $\tau$.

The result we want to emphasize is that

$$
\begin{align*}
t & =0 \Rightarrow \tau=0  \tag{23}\\
\tau & =0 \nRightarrow t=0
\end{align*}
$$

which implies that the imaging condition $\tau=0$ is more general than $t=0$. To image with $\tau=0$ means to consider only the up-down motion of the wave and disregard any horizontal displacement that it might have (see Figure 7). This way, headwaves from horizontal interfaces are regarded as a down-up motion, rather than the down-lateral-up, and the imaging condition seeks the point where the wave turns, i.e. it changes from a downward into an upward propagation.

## 6 Analytic example

In this section we describe an analytic example of imaging headwaves with constant velocity phase-shift migration but with $\tau=0$ imaging condition instead of $t=0$. The purpose of
the example is to show, as stated before, that the both headwaves from horizontal interfaces and reflections are imaged at the correct depth with $\tau=0$.
We consider a 3D acoustic experiment with source and receiver located at the same depth $(z=0)$ and one horizontal interface located at depth $z_{1}$ separating two media with wave propagation velocities $c_{0}$ and $c_{1}$. The media are assumed to have no lateral variation. The post-critical data in such an experiment is (see e.g. Aki and Richards Ch. 6)

$$
\begin{equation*}
P\left(x_{g}, y_{g}, 0 \mid x_{s}, y_{s}, 0 ; \omega\right)=P^{R}\left(x_{g}, y_{g}, 0 \mid x_{s}, y_{s}, 0 ; \omega\right)+P^{H}\left(x_{g}, y_{g}, 0 \mid x_{s}, y_{s}, 0 ; \omega\right) \tag{24}
\end{equation*}
$$

where

$$
\begin{equation*}
P^{R}\left(x_{g}, y_{g}, 0 \mid x_{s}, y_{s}, 0 ; \omega\right)=\frac{R}{d} \exp \left(i k_{r} r+i \nu_{0} 2 z_{1}\right) \tag{25}
\end{equation*}
$$

is the reflected event and

$$
\begin{equation*}
P^{H}\left(x_{g}, y_{g}, 0 \mid x_{s}, y_{s}, 0 ; \omega\right)=\frac{i}{\omega} \frac{c_{0}^{2}}{\left(1-c_{0}^{2} / c_{1}^{2}\right)} \frac{1}{r^{1 / 2} L^{3 / 2}} \exp \left(i \omega \frac{r}{c_{1}}+i \nu_{0} 2 z_{1}\right) \tag{26}
\end{equation*}
$$

is the headwave. In these expressions $\left(x_{s}, y_{s}, 0\right)$ and $\left(x_{g}, y_{g}, 0\right)$ are the positions of the source and receiver respectively, $\omega$ is the temporal frequency, $R$ is the angle-dependent reflection coefficient, $d$ is the total distance from the source to reflection point to receiver, $r$ is the horizontal offset and satisfies $r=\sqrt{\left(x_{g}-x_{s}\right)^{2}+\left(y_{g}-y_{s}\right)^{2}}, k_{r}$ is its conjugate in the K-space domain, $\nu_{0}$ is the vertical wavenumber of the first medium and satisfies $\nu_{0}^{2}+k_{r}^{2}=\omega^{2} / c_{0}^{2}$, and $L$ is the length of the horizontal part of the ray representation of the headwave.
We downward continue both the source and receiver to same arbitrary depth and obtain

$$
\begin{equation*}
P\left(x_{g}, y_{g}, z \mid x_{s}, y_{s}, z ; \omega\right)=P^{R}\left(x_{g}, y_{g}, z \mid x_{s}, y_{s}, z ; \omega\right)+P^{H}\left(x_{g}, y_{g}, z \mid x_{s}, y_{s}, z ; \omega\right) \tag{27}
\end{equation*}
$$

where

$$
\begin{equation*}
P^{R}\left(x_{g}, y_{g}, z \mid x_{s}, y_{s}, z ; \omega\right)=\frac{R}{d} \exp \left(i k_{r} r+i \nu_{0} 2\left(z_{1}-z\right)\right) \tag{28}
\end{equation*}
$$

and

$$
\begin{equation*}
P^{H}\left(x_{g}, y_{g}, z \mid x_{s}, y_{s}, z ; \omega\right)=\frac{i}{\omega} \frac{c_{0}^{2}}{\left(1-c_{0}^{2} / c_{1}^{2}\right)} \frac{1}{r^{1 / 2} L^{3 / 2}} \exp \left(i \omega \frac{r}{c_{1}}+i \nu_{0} 2\left(z_{1}-z\right)\right) \tag{29}
\end{equation*}
$$

The reflection can be easily imaged with the $t=0$ imaging condition to obtain the reflectivity of the reflection point at the correct depth $z_{1}$. The same procedure, applied to the part of the data representing the headwave, would assume that the turning point of that event is a point in space and it would seek that point hence creating an image at the wrong depth.
To image the headwave with $\tau=0$ we first inverse Fourier Transform to bring the data back to the time domain. However, $\omega$ is conjugated to the travel time $t$ and we want to bring the data back to the vertical time $\tau$ domain where we can apply the imaging condition. We define the image $I$ to be

$$
\begin{equation*}
I(\tau)=\int d \Omega e^{-i \Omega \tau} P^{H}\left(x_{g}, y_{g}, z \mid x_{s}, y_{s}, z ; \omega\right) \tag{30}
\end{equation*}
$$

where

$$
\begin{equation*}
\Omega=\omega\left(1+\frac{r}{c_{1} \tau}\right) . \tag{31}
\end{equation*}
$$

With the full expression for $P^{H}$ we have

$$
\begin{equation*}
I(\tau)=\int d \Omega e^{-i \Omega \tau} A(\omega) \exp \left(i \omega \frac{r}{c_{1}}+i \nu_{0} 2\left(z_{1}-z\right)\right) \tag{32}
\end{equation*}
$$

where $A(\omega)$ is the amplitude in equation (29). By plugging in the expression (31) for $\Omega$ we obtain

$$
\begin{equation*}
I(\tau)=\int d \Omega e^{-i \omega \tau} A(\omega) \exp \left(i \nu_{0} 2\left(z_{1}-z\right)\right) \tag{33}
\end{equation*}
$$

and after imaging with $\tau=0$ we find

$$
\begin{equation*}
I(\tau=0)=\int d \Omega A(\omega) \exp \left(i \nu_{0} 2\left(z_{1}-z\right)\right) \tag{34}
\end{equation*}
$$

This last expression represents a delta like event at the correct depth $z_{1}$ hence showing that the headwave is imaged correctly.

Notice that the new condition discards any horizontal propagation (and time associated with it) and only takes into consideration down-up propagations, as one can also see from Figure 7. For the headwave this means discarding the horizontal propagation along the interface. It is not difficult to see that the procedure outlined above also images the reflection data at the correct depth.

## 7 Conclusions

The purpose of this paper is to present the advantages of imaging with $\tau=0$ versus imaging with $t=0$. The former condition is a generalization of the latter which has the ability to image headwaves from horizontal interfaces at the correct depth. This suggests that using $\tau=0$ in the internal multiples attenuation algorithm derived from inverse scattering series techniques would generalize it to address internal multiples constructed with this kind of sub-events. This research is part of our strategy to include more of the returning signal, information bearing or not, in our analysis and processing.

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# Linear inversion of absorptive/dispersive wave field measurements: theory and 1D synthetic tests 

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#### Abstract

The use of inverse scattering theory for the inversion of viscoacoustic wave field measurements, namely for a set of parameters that includes $Q$, is by its nature very different from most current approaches for $Q$ estimation. In particular, it involves an analysis of the angle- and frequency-dependence of amplitudes of viscoacoustic data events, rather than the measurement of temporal changes in the spectral nature of events. We consider the linear inversion for these parameters theoretically and with synthetic tests. The output is expected to be useful in two ways: (1) on its own it provides an approximate distribution of $Q$ with depth, and (2) higher order terms in the inverse scattering series as it would be developed for the viscoacoustic case would take the linear inverse as input.

We will begin, following Innanen (2003) by casting and manipulating the linear inversion problem to deal with absorption for a problem with arbitrary variation of wavespeed and $Q$ in depth, given a single shot record as input. Having done this, we will numerically and analytically develop a simplified instance of the 1D problem. This simplified case will be instructive in a number of ways, first of all in demonstrating that this type of direct inversion technique relies on reflectivity, and has no interest in or ability to analyse propagation effects as a means to estimate $Q$. Secondly, through a set of examples of slightly increasing complexity, we will demonstrate how and where the linear approximation causes more than the usual levels of error. We show how these errors may be mitigated through use of specific frequencies in the input data, or, alternatively, through a layer-stripping based, or bootstrap, correction. In either case the linear results are encouraging, and suggest the viscoacoustic inverse Born approximation may have value as a standalone inversion procedure.


## 1 Introduction

A well-known and oft-mentioned truism in reflection seismic data processing is that, broadly put, the velocity structure of the subsurface impacts the recorded wave field in two important ways:
(1) rapid variations in Earth properties (such as velocity) give rise to reflection effects, and
(2) slow variations give rise to propagation effects (e.g. move-out etc.)

Wave theory predicts an exact parallel of this truism for the case of an absorptive/dispersive medium. That is,
(1) contrasts in absorptive/dispersive Earth parameters produce a characteristic reflectivity, and
(2) trends cause characteristic propagation effects, namely amplitude decay and dispersion.

In spite of this direct parallel, there is a striking discrepancy between the way seismic parameter inversion takes place in these two instances; acoustic/elastic inversion makes primary use of (1), via AVO-like methods, and absorptive/dispersive inversion makes use of (2), usually via the study of trends in amplitude decay for $Q$ estimation. The reason for the discrepancy is entirely practical: for absorptive/dispersive media, the propagation effects of $Q$ dominate over the reflectivity effects. It is certainly very sensible to make use of the dominant effects of a parameter in its estimation (see for instance Tonn, 1991; Dasgupta and Clark, 1998). Notwithstanding, permit us to make some comments negative to this approach.

First, a correction: of course, acoustic data processing does involve propagation-based inversion, in velocity analysis (but the output velocity field is not considered an end in itself). This will be an instructive analogy. Both propagation-based velocity analysis, and propagationbased $Q$-estimation, gain their effectiveness by evaluating changes that span the data set (in spatial and temporal domains), either by monitoring move-out or by monitoring ratios of spectral amplitudes. Estimating parameters by observing trends in the data set must be a somewhat ad hoc process, always requiring some level of assumption about the nature of the medium. Examples are so well-known as to be scarcely worth mentioning, but one thinks immediately of NMO-based velocity analysis, which in its most basic form requires a medium made up of horizontal layers. The difference between $Q$-estimation techniques and velocity analysis techniques is that the latter have been developed to states of great complexity and sophistication, such that many of these destructive assumptions are avoided (e.g., through techniques of reflection tomography). Comparatively, most $Q$ estimation techniques are simple, often based on the assumption that there is a single $Q$ value that dictates the absorptive behaviour of a wave field everywhere in the medium.

If we seriously think that the data we measure are shaped and altered by wave propagation that follows a known attenuation law, and if we want to be able to determine the medium parameters, including $Q$, badly enough to (a) take high quality data and (b) look at it very closely, then an increased level of sophistication is required.

An effort to usefully increase the level of sophistication of $Q$ estimation can go one of two or more ways (we'll mention two), and this harkens back to the aforementioned discrepancy in inversion approaches. First, we could follow the development of propagation based inversion, or velocity analysis, towards a tomographic/ray tracing milieu in which local spectral characteristics of an event are permitted to be due to a $Q$ that varies along the ray path, and a spatial distribution of $Q$ is estimated along these lines. This could provide a useful, but smooth, spatial distribution of $Q$.

Second, we could make absorptive/dispersive inversion procedures more closely imitate their
acoustic/elastic brethren, and focus rather on a close analysis of angle- and frequencydependent amplitudes of data events. There are many reasons to shy away from this kind of approach, all of which stem from the idea of dominant effects - $Q$-like reflectivity is a lot less detectable than $Q$-like propagation.

The reasons in favour of the pursuit of such an inversion for absorptive/dispersive medium parameters likewise all stem from a single idea: the inverse scattering series demands that we do it that way. We listen to such demands because of the promise of the inverse scattering series: to provide a multidimensional reconstruction of the medium parameters that gave rise to the scattered wave field, with no assumptions about the structure of the medium, and no requirement of an accurate velocity model as input. Suppose we measure the scattered wave field above an absorptive/dispersive medium with sharp contrasts in $Q$ as well as the wavespeed, and suppose we cast the inverse scattering series problem with an acoustic (non-attenuating) Green's function. First, since the series will reconstruct the sharp medium transitions from attenuated - smoothed - data, a de facto $Q$ compensation must be occurring. Second, since $Q$ is entirely within the perturbation (given an acoustic reference), the reconstruction is a de facto $Q$ estimation. In other words, without dampening our spirits by considering issues of practical implementation, the viscoacoustic inverse scattering series must accomplish these two tasks, a multidimensional $Q$ compensation and estimation, in the absence of an accurate foreknowledge of $Q$. It is this promise that motivates an investigation into the use of inverse scattering techniques to process and invert absorptive/dispersive wave field measurements.

The first step in doing so is to investigate the linear inversion problem, and it is to this component of the problem that the bulk of this paper is geared. The results of linear inversion are of course often tremendously useful on their own, and this is both true and untrue of the absorptive/dispersive case.

We will begin by casting and manipulating the linear inversion to deal with arbitrary variation of wavespeed and $Q$ in depth, given a single shot record as input. Having done this, we will numerically and analytically develop a simplified instance of the 1D problem. This simplified case will be instructive in a number of ways, first of all in demonstrating that this type of direct inversion technique relies on reflectivity, and has no interest in or ability to analyze propagation effects as a means to estimate $Q$. Secondly, through a set of examples of slightly increasing complexity, we will demonstrate how and where the linear approximation causes more than the usual levels of error. We show how these errors may be mitigated through use of specific frequencies in the input data, or, alternatively, through a layer-stripping based, or bootstrap, correction. In either case the linear results are encouraging, and suggest the viscoacoustic Born approximation may have value as a standalone inversion procedure.

Obviously analysis of this kind relies heavily on correctly modelling the behaviour of the reflection coefficient at viscous boundaries. We give this important question short shrift here, by taking a well-known model for attenuation and swallowing it whole; and to be sure, the quality of the inversion results depend on the adequacy of these models to predict the behaviour of the viscous reflection coefficient. On the other hand, the frequency dependence
of $R(f)$, which provides the information driving the inversion, is a consequence of contrasts in media with dispersive behaviour. All theory falls in line given the presence of a dispersive character in the medium, in principle if not in the detail of this chosen attenuation model.

## 2 Casting the Absorptive/Dispersive Problem

In acoustic/elastic/anelastic (etc.) wave theory, the parameters describing a medium are related non-linearly to the measurements of the wave field. Many forms of direct wave field inversion, including those used in this paper, involve a linearization of the problem, in other words a solution for those components of the model which are linear in the measured data.

There are two reasons for solving for the linear portion of the model. First, if the reference Green's function is sufficiently close to the true medium, then the linear portion of the model may be, in and of itself, of value as a close approximation to the true Earth. Second, a particular casting of the inverse scattering series uses this linear portion of the scattering potential (or model) as input for the solution of higher order terms. It is useful to bear in mind that the decay of the proximity of the Born inverse to the real Earth does not, in methods based on inverse scattering, signal the end of the utility of the output. Rather, it marks the start of the necessity for inclusion, if possible, of higher order terms - terms "beyond Born".

Seismic events are often better modelled as having been generated by changes in multiple Earth parameters than in a single one; for instance, density and wavespeed in an impedancetype description, or density and bulk modulus in a continuum mechanics-type description. In either case, the idea is that a single parameter velocity inversion (after that of Cohen and Bleistein (1977)) encounters problems because the amplitude of events is not reasonably explicable with a single parameter.

In Clayton and Stolt (1981) and Raz (1981), density/bulk modulus and density/wavespeed models respectively are used with a single-scatterer approximation to invert linearly for profiles of these parameters. In both cases it is the variability of the data in the offset dimension that provides the information necessary to separate the two parameters. The key (Clayton and Stolt, 1981; Weglein, 1985) is to arrive at a relationship between the data and the linear model components in which, for each instance of an experimental variable, an independent equation is produced. For instance, in an AVO type problem, an overdetermined system of linear equations is produced (one equation for each offset), which may be solved for multiple parameters.

In a physical problem involving dispersion, waves travel at different speeds depending on the frequency, which means that, at regions of sharp change of the inherent viscoacoustic properties of the medium, frequency-dependent reflection coefficients are found. This suggests that one might look to the frequency content of the data as a means to similarly separate some appropriately-chosen viscoacoustic parameters (i.e. wavespeed and $Q$ ).

We proceed by adopting a $Q$ model similar to those discussed by Aki and Richards (2002) and equivalent to that of Kjartansson (1979) under certain assumptions, such that the dispersion relation is assumed, over a reasonable seismic bandwidth, to be given by

$$
\begin{equation*}
k(z)=\frac{\omega}{c(z)}\left[1+\frac{i}{2 Q(z)}-\frac{1}{\pi Q(z)} \ln \left(\frac{k}{k_{r}}\right)\right] \tag{1}
\end{equation*}
$$

where $k_{r}=\omega_{r} / c_{0}$ is a reference wavenumber, $k=\omega / c_{0}$, and where $c_{0}$ is a reference wavespeed to be discussed presently. As discussed previously, this specific choice of $Q$ model is crucial to the mathematical detail of what is to follow; however we consider the general properties of the inverse method we develop to be well geared to handle the general properties of the $Q$ model.
This is re-writeable using an attenuation parameter $\beta(z)=1 / Q(z)$ multiplied by a function $F(k)$, of known form:

$$
\begin{equation*}
F(k)=\frac{i}{2}-\frac{1}{\pi} \ln \left(\frac{k}{k_{r}}\right), \tag{2}
\end{equation*}
$$

which utilizes $\beta(z)$ to correctly instill both the attenuation $(i / 2)$ and dispersion $\left(-\frac{1}{\pi} \ln \left(k / k_{r}\right)\right)$. Notice that $F(k)$ is frequency-dependent because of the dispersion term. Then

$$
\begin{equation*}
k(z)=\frac{\omega}{c(z)}[1+\beta(z) F(k)] \tag{3}
\end{equation*}
$$

The linearized Born inversion is based on a choice for the form of the scattering potential $V$, which is given by

$$
\begin{equation*}
\mathbf{V}=\mathbf{L}-\mathbf{L}_{0} \tag{4}
\end{equation*}
$$

or the difference of the wave operators describing propagation in the reference medium $\left(\mathbf{L}_{0}\right)$ and the true medium $(\mathbf{L})$. For a constant density medium with a homogeneous reference this amounts to

$$
\begin{equation*}
\mathbf{V}=V(x, z, k)=k^{2}(x, z)-\frac{\omega^{2}}{c_{0}^{2}}, \tag{5}
\end{equation*}
$$

for a medium which varies in two dimensions, or

$$
\begin{equation*}
V(z, k)=k^{2}(z)-\frac{\omega^{2}}{c_{0}^{2}}, \tag{6}
\end{equation*}
$$

for a 1D profile. Using equation (3), we specify the wavespeed/ $Q$ scattering potential to be

$$
\begin{equation*}
V(x, z, k)=\frac{\omega^{2}}{c^{2}(x, z)}[1+\beta(x, z) F(k)]^{2}-\frac{\omega^{2}}{c_{0}^{2}} \tag{7}
\end{equation*}
$$

and include the standard perturbation on the wavespeed profile $c(x, z)$ in terms of $\alpha(x, z)$ and a reference wavespeed $c_{0}$, producing

$$
\begin{align*}
V(x, z, k) & =\frac{\omega^{2}}{c_{0}^{2}}[1-\alpha(x, z)][1+\beta(x, z) F(k)]^{2}-\frac{\omega^{2}}{c_{0}^{2}} \\
& \approx-\frac{\omega^{2}}{c_{0}^{2}}[\alpha(x, z)-2 \beta(x, z) F(k)], \tag{8}
\end{align*}
$$

dropping all terms quadratic and higher in the perturbations $\alpha$ and $\beta$. The 1D profile version of this scattering potential is then, straightforwardly

$$
\begin{equation*}
V(z, k) \approx-\frac{\omega^{2}}{c_{0}^{2}}[\alpha(z)-2 \beta(z) F(k)] \tag{9}
\end{equation*}
$$

The scattering potential in equation (9) will be used regularly in this paper.

## 3 Inversion for $Q /$ Wavespeed Variations in Depth

The estimation of the 1D contrast (i.e. in depth) of multiple parameters from seismic reflection data is considered, similar to, for instance, Clayton and Stolt (1981). For the sake of exposition we demonstrate how the problem is given the simplicity of a normal-incidence experiment by considering the bilinear form of the Green's function. In 1D, for instance, the Green's function, which has the nominal form

$$
\begin{equation*}
G_{0}\left(z_{g} \mid z^{\prime} ; \omega\right)=\frac{e^{i k\left|z_{g}-z^{\prime}\right|}}{2 i k} \tag{10}
\end{equation*}
$$

also has the bilinear form

$$
\begin{equation*}
G_{0}\left(z_{g} \mid z^{\prime} ; \omega\right)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} d k_{z}^{\prime} \frac{e^{i k_{z}^{\prime}\left(z_{g}-z^{\prime}\right)}}{k^{2}-k_{z}^{\prime 2}} \tag{11}
\end{equation*}
$$

where $k=\omega / c_{0}$. Consider the reference medium to be acoustic with constant wavespeed $c_{0}$. The scattered wave field (measured at $x_{g}, z_{g}$ for a source at $\left.x_{s}, z_{s}\right), \psi_{s}\left(x_{g}, z_{g} \mid x_{s}, z_{s} ; \omega\right.$ ), is related to model components that are linear in the data; these are denoted $V_{1}(z, \omega)$. This relationship is given by the exact equation

$$
\begin{equation*}
\psi_{s}\left(x_{g}, z_{g} \mid x_{s}, z_{s} ; \omega\right)=S(\omega) \int_{-\infty}^{\infty} d x^{\prime} \int_{-\infty}^{\infty} d z^{\prime} G_{0}\left(x_{g}, z_{g} \mid x^{\prime}, z^{\prime} ; \omega\right) V_{1}\left(z^{\prime}, \omega\right) G_{0}\left(x^{\prime}, z^{\prime} \mid x_{s}, z_{s} ; \omega\right) \tag{12}
\end{equation*}
$$

where $S$ is the source waveform. The function $G_{0}$ describes propagation in the acoustic reference medium, and can be written as a 2D Green's function in bilinear form:

$$
\begin{equation*}
G_{0}\left(x_{g}, z_{g} \mid x^{\prime}, z^{\prime} ; \omega\right)=\frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} d k_{x}^{\prime} \int_{-\infty}^{\infty} d k_{z}^{\prime} \frac{e^{i k_{x}^{\prime}\left(x_{g}-x^{\prime}\right)} e^{i k_{z}^{\prime}\left(z_{g}-z^{\prime}\right)}}{k^{2}-\left(k_{x}^{\prime 2}+k_{z}^{\prime 2}\right)}, \tag{13}
\end{equation*}
$$

where $k=\omega / c_{0}$. Measurements over a range of $x_{g}$ will permit a Fourier transform to the coordinate $k_{x g}$ in the scattered wave field. On the right hand side of equation (12) this amounts to taking the Fourier transform of the left Green's function $G_{0}\left(x_{g}, z_{g} \mid x^{\prime}, z^{\prime} ; \omega\right)$ :

$$
\begin{equation*}
G_{0}\left(k_{x g}, z_{g} \mid x^{\prime}, z^{\prime} ; \omega\right)=\frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} d k_{x}^{\prime} \int_{-\infty}^{\infty} d k_{z}^{\prime} \int_{-\infty}^{\infty} d x_{g} \frac{e^{-i k_{x g} x_{g}} e^{i k_{x}^{\prime}\left(x_{g}-x^{\prime}\right)} e^{i k_{z}^{\prime}\left(z_{g}-z^{\prime}\right)}}{k^{2}-\left(k_{x}^{\prime 2}+k_{z}^{\prime 2}\right)} . \tag{14}
\end{equation*}
$$

Taking advantage of the sifting property of the Fourier transform:

$$
\begin{align*}
G_{0}\left(k_{x g}, z_{g} \mid x^{\prime}, z^{\prime} ; \omega\right) & =\frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} d k_{x}^{\prime} \int_{-\infty}^{\infty} d k_{z}^{\prime} \int_{-\infty}^{\infty} d x_{g} \frac{e^{i\left(k_{x g}-k_{m}^{\prime}\right) x_{g}} e^{-i k_{x}^{\prime} x^{\prime}} e^{i k_{z}^{\prime}\left(z_{g}-z^{\prime}\right)}}{k^{2}-\left(k_{x}^{\prime 2}+k_{z}^{\prime 2}\right)} \\
& =\frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} d k_{x}^{\prime} \int_{-\infty}^{\infty} d k_{z}^{\prime} \frac{e^{-i k_{x}^{\prime} x^{\prime}} e^{i k_{z}^{\prime}\left(z_{g}-z^{\prime}\right)}}{k^{2}-\left(k_{x}^{\prime 2}+k_{z}^{\prime 2}\right)}\left[\int_{-\infty}^{\infty} d x_{g} e^{i\left(k_{x g}-k_{m}^{\prime}\right) x_{g}}\right] \\
& =\frac{1}{2 \pi} \int_{-\infty}^{\infty} d k_{x}^{\prime} \int_{-\infty}^{\infty} d k_{z}^{\prime} \frac{e^{-i k_{x}^{\prime} x^{\prime}} e^{i k_{z}^{\prime}\left(z_{g}-z^{\prime}\right)}}{k^{2}-\left(k_{x}^{\prime 2}+k_{z}^{\prime 2}\right)} \delta\left(k_{x}^{\prime}-k_{x g}\right)  \tag{15}\\
& =\frac{1}{2 \pi} e^{-i k_{x g} x^{\prime}} \int_{-\infty}^{\infty} d k_{z}^{\prime} \frac{e^{i k_{z}^{\prime}\left(z_{g}-z^{\prime}\right)}}{q_{z}^{2}+k_{z}^{\prime 2}},
\end{align*}
$$

where $q_{z}^{2}=k^{2}-k_{x g}{ }^{2}$, a vertical wavenumber. Notice that the remaining integral is a 1D Green's function in bilinear form, as in equation (11). So equation (15) takes on the remarkably simplified form:

$$
\begin{equation*}
G_{0}\left(k_{x g}, z_{g} \mid x^{\prime}, z^{\prime} ; \omega\right)=e^{-i k_{x g} x^{\prime}}\left[\frac{e^{i q_{z}\left|z_{g}-z^{\prime}\right|}}{2 i q_{z}}\right] . \tag{16}
\end{equation*}
$$

The righthand Green's function in equation (12) may likewise be written

$$
\begin{equation*}
G_{0}\left(x^{\prime}, z^{\prime} \mid x_{s}, z_{s} ; \omega\right)=\frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} d k_{x s} \int_{-\infty}^{\infty} d k_{z s} \frac{e^{i k_{x s}\left(x^{\prime}-x_{s}\right)} e^{i k_{z s}\left(z^{\prime}-z_{s}\right)}}{k^{2}-\left(k_{x s}^{2}+k_{z s}^{2}\right)}, \tag{17}
\end{equation*}
$$

and therefore the scattered wave field becomes

$$
\begin{align*}
\psi_{s}\left(k_{x g}, z_{g} \mid x_{s}, z_{s} ; \omega\right)= & S(\omega) \frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} d x^{\prime} \int_{-\infty}^{\infty} d z^{\prime} e^{-i k_{x s} x^{\prime}} \frac{e^{i q_{z}\left|z_{g}-z^{\prime}\right|}}{2 i q_{z}} \times \\
& \int_{-\infty}^{\infty} d k_{x s} \int_{-\infty}^{\infty} d k_{z s} \frac{e^{i k_{x s}\left(x^{\prime}-x_{s}\right)} e^{i k_{z s}\left(z^{\prime}-z_{s}\right)}}{k^{2}-\left(k_{x s}{ }^{2}+k_{z s}{ }^{2}\right)} V_{1}\left(z^{\prime}, \omega\right) . \tag{18}
\end{align*}
$$

The seismic experiment is conducted along a surface, which for convenience may be set at $z_{s}=z_{g}=0$. Further, since the subsurface being considered has variation in $z$ only, all "shot-record" type experiments are identical, and only one need be considered. We let this one shot be at $x_{s}=0$. This produces the simplified expression

$$
\begin{align*}
\psi_{s}\left(k_{x g}, 0 \mid 0,0 ; \omega\right)= & S(\omega) \frac{1}{(2 \pi)^{2}} \int_{-\infty}^{\infty} d x^{\prime} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d k_{x s} \int_{-\infty}^{\infty} d k_{z s} \frac{e^{i q_{z} z^{\prime}}}{2 i q_{z}} \times  \tag{19}\\
& \frac{e^{i k_{z s} z^{\prime}}}{k^{2}-\left(k_{x s}^{2}+k_{z s}^{2}\right)} e^{i\left(k_{x s}-k_{x g}\right) x^{\prime}} V_{1}\left(z^{\prime}, \omega\right),
\end{align*}
$$

which, similarly to equation (15), becomes

$$
\begin{align*}
\psi_{s}\left(k_{x g}, 0 \mid 0,0 ; \omega\right) & =S(\omega) \frac{1}{2 \pi} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d k_{x s} \int_{-\infty}^{\infty} d k_{z s} \frac{e^{i q_{z} z^{\prime}}}{2 i q_{z}} \frac{e^{i k_{z s} z^{\prime}}}{k^{2}-\left(k_{x s}{ }^{2}+k_{z s}{ }^{2}\right)} \delta\left(k_{x s}-k_{x g}\right) V_{1}\left(z^{\prime}, \omega\right) \\
& =S(\omega) \frac{1}{2 \pi} \int_{-\infty}^{\infty} d z^{\prime} \int_{-\infty}^{\infty} d k_{z s} \frac{e^{i q_{z} z^{\prime}}}{2 i q_{z}} \frac{e^{i k_{z s} z^{\prime}}}{k^{2}-\left(k_{x g}{ }^{2}+k_{z s}{ }^{2}\right)} V_{1}\left(z^{\prime}, \omega\right) \\
& =S(\omega) \frac{1}{2 \pi} \int_{-\infty}^{\infty} d z^{\prime} \frac{e^{i q_{z} z^{\prime}}}{2 i q_{z}} V_{1}\left(z^{\prime}, \omega\right)\left[\int_{-\infty}^{\infty} d k_{z s} \frac{e^{i k_{z s} z^{\prime}}}{q_{z}^{2}+k_{z s}^{2}}\right] \tag{20}
\end{align*}
$$

where again the vertical wavenumber $q_{z}^{2}=k^{2}-k_{x g}^{2}$ appears. The integral over $d k_{z s}$ has the form of a 1D Green's function. The data equations (one for each frequency), with the choice $z_{s}=z_{g}=x_{s}=0$, are now

$$
\begin{align*}
\psi_{s}\left(k_{x g} ; \omega\right) & =S(\omega) \int_{-\infty}^{\infty} d z^{\prime} \frac{e^{i q_{z} z^{\prime}}}{2 i q_{z}} V_{1}\left(z^{\prime}, \omega\right) \frac{e^{i q_{z} z^{\prime}}}{2 i q_{z}} \\
& =-\frac{S(\omega)}{4 q_{z}^{2}} \int_{-\infty}^{\infty} d z^{\prime} e^{i 2 q_{z} z^{\prime}} V_{1}\left(z^{\prime}, \omega\right)  \tag{21}\\
& =-\frac{S(\omega)}{4 q_{z}^{2}} V_{1}\left(-2 q_{z}, \omega\right),
\end{align*}
$$

recognizing that the last integral is a Fourier transform of the scattering potential $V_{1}$. Thus one has an expression of the unknown perturbation $V_{1}$ (i.e. the model) that is linear in the data. Multiple parameters within $V_{1}$ may be solved if, frequency by frequency, the data equations (21) are independent.

Estimation of multiple parameters from data with offset (i.e. using AVO) requires that equations (21) be independent offset to offset. In the case of viscoacoustic inversion, we show that it is the dispersive nature of an attenuation model which produces the independence of the data equations with offset, allowing the procedure to go forward.

Consider the term $F(k)$ :

$$
\begin{equation*}
F(k)=\frac{i}{2}-\frac{1}{\pi} \ln \left(\frac{k}{k_{r}}\right) . \tag{22}
\end{equation*}
$$

As mentioned, the frequency dependence of $F$ arises from the rightmost component in equation (22), the dispersion component. In 1D wave propagation, this amounts to the "rule" by which the speed of the wave field alters, frequency by frequency, with respect to the reference wavenumber $k_{r}=\omega_{r} / c_{0}$, usually chosen using the largest frequency of the seismic experiment. In 2D wave propagation, $F$, which changes the propagation wavenumber $k(z)$ in equation (3), now alters the wave field along its direction of propagation in $(x, z)$. Let $\theta$ represent the angle away from the downward, positive, $z$ axis. A vertical wavenumber $q_{z}$ is related to $k$ by $q_{z}=k \cos \theta$; if one replaces the reference wavenumber $k_{r}$ with a reference angle $\theta_{r}$ and reference vertical wavenumber $q_{z r}$, then $F$ becomes

$$
\begin{align*}
F(k) & =\frac{i}{2}-\frac{1}{\pi} \ln \left(\frac{k}{k_{r}}\right) \\
& =\frac{i}{2}-\frac{1}{\pi} \ln \left(\frac{q_{z} \cos \theta_{r}}{q_{z r} \cos \theta}\right) . \tag{23}
\end{align*}
$$

Then:

$$
\begin{equation*}
F\left(\theta, q_{z}\right)=\frac{i}{2}-\frac{1}{\pi} \ln \left(\frac{q_{z} \cos \theta_{r}}{q_{z r} \cos \theta}\right), \tag{24}
\end{equation*}
$$

so what remains is a function which, for a given vertical wavenumber, predicts an angle dependent alteration to the wave propagation. As such the scattering potential may be written as a function of angle and vertical wavenumber also:

$$
\begin{equation*}
V\left(z, \theta, q_{z}\right)=-\frac{\omega^{2}}{c_{0}^{2}}\left[\alpha(z)-2 \beta(z) F\left(\theta, q_{z}\right)\right] . \tag{25}
\end{equation*}
$$

The angle dependence of $F$ produces independent sets of data equations, since it alters the coefficient of $\beta(z)$ for different angles while leaving $\alpha(z)$ untouched. Using equation (25), one may write the linear component of a depth-dependent only scattering potential as:

$$
\begin{equation*}
V_{1}\left(z, \theta, q_{z}\right)=-\frac{\omega^{2}}{c_{0}^{2}}\left[\alpha_{1}(z)-2 \beta_{1}(z) F\left(\theta, q_{z}\right)\right] . \tag{26}
\end{equation*}
$$

Recall from earlier in this section that the requisite data equations are

$$
\begin{equation*}
\psi_{s}\left(k_{x g} ; \omega\right)=-\frac{S(\omega)}{4 q_{z}^{2}} V_{1}\left(-2 q_{z}, \omega\right) ; \tag{27}
\end{equation*}
$$

using equation (26), and considering the surface expression of the wave field to be the data, deconvolved of $S(\omega)$, this becomes

$$
\begin{equation*}
D\left(q_{z}, \theta\right)=K_{1}(\theta) \alpha_{1}\left(-2 q_{z}\right)+K_{2}\left(\theta, q_{z}\right) \beta_{1}\left(-2 q_{z}\right), \tag{28}
\end{equation*}
$$

where

$$
\begin{equation*}
K_{1}(\theta)=\frac{1}{4 \cos ^{2} \theta}, \quad K_{2}\left(\theta, q_{z}\right)=-2 \frac{F\left(\theta, q_{z}\right)}{4 \cos ^{2} \theta} . \tag{29}
\end{equation*}
$$

Notice that in equation (29) we have more than one equation at each wavenumber $q_{z}$; every offset or angle $\theta$ provides an independent equation, and so in an experiment with many offsets we have an overdetermined problem.

## 4 A Complex, Frequency Dependent Reflection Coefficient

The success of such an attempt to extract linear viscoacoustic perturbations as above is obviously, therefore, contingent on detecting the impact of the frequency-dependent viscoacoustic reflection coefficient $R(k)$ on the data amplitudes. This is an important aspect of a scattering-based attempt to process and invert seismic data taking such lossy propagation into account: the inverse scattering series will look to the frequency- and angle-dependent aspects of the measured events for information on $Q$.

Using previously-defined terminology, the reflection coefficient for an 1D acoustic wave field normally incident on a contrast in wavespeed (from $c_{0}$ to $c_{1}$ ) and $Q$ (from $\infty$ to $Q_{1}$ ), is

$$
\begin{equation*}
R(k)=\frac{1-\frac{c_{0}}{c_{1}}\left(1+\frac{F(k)}{Q_{1}}\right)}{1+\frac{c_{0}}{c_{1}}\left(1+\frac{F(k)}{Q_{1}}\right)} \tag{30}
\end{equation*}
$$

This is a complex, frequency ${ }^{1}$ dependent quantity that will alter the amplitude and phase spectra of the measured wave field. The spectra of reflection coefficients of this form for a single wavespeed contrast ( $c_{0}=1500 \mathrm{~m} / \mathrm{s}$ to $c_{1}=1600 \mathrm{~m} / \mathrm{s}$ ) and a variety of $Q_{1}$ values is illustrated in Figure 1. The attenuative reflection coefficient approaches its acoustic counterpart as $Q_{1} \rightarrow \infty$; the variability of $R$ with $f$ increases away from the reference wavenumber. Equation (1), and hence equation (30), relies on $\left|\frac{1}{Q_{1}} \ln \left(\frac{f}{f_{r}}\right)\right| \ll 1$, and so at low frequency we must consider the accuracy of the current $Q$ model. But a problematic $\ln \left(f / f_{r}\right) \approx-Q_{1}$ requires $f / f_{r} \approx e^{-Q_{1}}$, which keeps us out of trouble for almost all realistic combinations of $f$ and $Q$, with the exception of the very lowest frequencies.

[^1]

Figure 1: Real component of the reflection coefficient $R(f)$, where $f=k_{0} / 2 \pi$, over the frequency interval associated with a $4 s$ experiment with $\Delta t=0.004 s$, and a reference frequency of $k_{r} c_{0} / 2 \pi=f_{r}=125 \mathrm{~Hz}$. The acoustic (non-attenuating) reflection coefficient $R^{\prime}=\left(c_{0}-c_{1}\right) /\left(c_{0}+c_{1}\right)$ is included as a dashed line. $Q_{1}$ values are (a) 1000, (b) 500, (c) 100, and (d) 50.

## 5 Analytic/Numeric Tests: The 1D Normal Incidence Problem

In general it is not possible to invert for two parameters from a 1D normal incidence seismic experiment. However, if one assumes a basic spatial form for the Earth model (or perturbation from reference model), then this problem becomes tractable for a dispersive Earth. The discussion in this section continues along these lines, i.e. diverging from the more general inversion formalism developed previously. In doing so, it benefits from the simplicity of the 1 D normal incidence example: many key features of the "normal incidence + structural assumptions" problem are shared by the "offset + no structural assumptions" problem, but the former are easier to compute and analyze.

Consider an experiment with coincident source and receiver $z_{s}=z_{g}=0$. The linear data equation, in which the data are assumed to be the scattered field $\psi_{s}$ measured at this source/receiver point, is

$$
\begin{equation*}
D(k)=\psi_{s}(0 \mid 0 ; k)=\int_{-\infty}^{\infty} G_{0}\left(0 \mid z^{\prime} ; k\right) k^{2} \gamma_{1}\left(z^{\prime}\right) \psi_{0}\left(z^{\prime} \mid 0 ; k\right) d z^{\prime} \tag{31}
\end{equation*}
$$

which, following the substitution of the acoustic 1D homogeneous Green's function and plane
wave expressions $G_{0}$ and $\psi_{0}$ becomes

$$
\begin{equation*}
D(k)=-\frac{1}{2} i k \gamma_{1}(-2 k) \tag{32}
\end{equation*}
$$

where the integral is recognized as being a Fourier transform of the linear portion of the perturbation, called $\gamma_{1}(z)$. The form for the perturbation is given by the difference between the wave operators for the reference medium $\left(\mathbf{L}_{0}\right)$ and the non-reference medium ( $\mathbf{L}$ ), as discussed above. In this case, let the full scattering potential be due to a perturbation $\gamma$ :

$$
\begin{equation*}
\gamma(z)=\frac{V(z, k)}{k^{2}} \tag{33}
\end{equation*}
$$

where $V(z, k)$ is given by equation (9). Writing the linear portion of the overall perturbation as

$$
\begin{equation*}
\gamma_{1}(z)=2 \beta_{1}(z) F(k)-\alpha_{1}(z), \tag{34}
\end{equation*}
$$

taking its Fourier transform, and inserting it into equation (32), the data equations

$$
\begin{equation*}
D(k)=-\frac{1}{2} i k\left[2 \beta_{1}(-2 k) F(k)-\alpha_{1}(-2 k)\right], \tag{35}
\end{equation*}
$$

or

$$
\begin{equation*}
\alpha_{1}(-2 k)-2 \beta_{1}(-2 k) F(k)=4 \frac{D(k)}{i 2 k} \tag{36}
\end{equation*}
$$

are produced.
Equation (36) as it stands cannot be used to separate $\alpha_{1}$ and $\beta_{1}$. This is because at every wavenumber one has a single equation and two unknowns. However, much of the information garnered from the data, frequency by frequency, is concerned with determining the spatial distribution of these parameters. If a specific spatial dependence is imposed on $\alpha_{1}$ and $\beta_{1}$ the situation is different.

Consider a constant density acoustic reference medium (a 1D homogeneous whole space) characterized by wavespeed $c_{0}$; let it be perturbed by a homogeneous viscoacoustic halfspace, characterized by the wavespeed $c_{1}$, and now also by the $Q$-factor $Q_{1}$. The contrast occurs at $z=z_{1}>0$. Physically, this configuration amounts to probing a step-like interface with a normal incidence wave field, in which the medium above the interface (i.e. the acoustic overburden) is known. This is illustrated in Figure 2.

Data from an experiment over such a configuration are measurements of a wave field event that has a delay of $2 z_{1} / c_{0}$ and that is weighted by a complex, frequency dependent reflection coefficient:


Figure 2: Single interface experiment involving a contrast in wavespeed $c_{0}$ and $Q$.

$$
\begin{equation*}
D(k)=R(k) e^{i 2 k z_{1}} \tag{37}
\end{equation*}
$$

This may be equated to the right-hand side of equation (35), in which the perturbation parameters are given the spatial form of a Heaviside function with a step at $z_{1}$. This is the pseudo-depth, or the depth associated with the reference wavespeed $c_{0}$ and the measured arrival time of the reflection. The data equations become

$$
\begin{equation*}
R(k) e^{i 2 k z_{1}}=\frac{1}{2} i k\left[\alpha_{1} \frac{e^{i 2 k z_{1}}}{i 2 k}-2 \beta_{1} \frac{e^{i 2 k z_{1}}}{i 2 k} F(k)\right], \tag{38}
\end{equation*}
$$

or

$$
\begin{equation*}
\alpha_{1}-2 \beta_{1} F(k)=4 R(k), \tag{39}
\end{equation*}
$$

in which $\alpha_{1}$ and $\beta_{1}$ are constants. So having assumed a spatial form for the perturbations, the data equations (39) are now overdetermined, with two unknowns and as many equations as there are frequencies in the experiment. Notice that it is the frequency dependence of $F(k)$ that ensures these equations are independent - hence, it is the dispersive nature of the attenuative medium that permits the inversion to take place.

In an experiment with some reasonable bandwidth, the above relationship constitutes an overdetermined problem. If the reflection coefficient at each of $N$ available frequencies $\omega_{n}$ (in which we label $k_{n}=\omega_{n} / c_{0}$ ) are the elements of a column vector $\mathbf{R}$, the unknowns $\alpha_{1}$ and $\beta_{1}$ are the elements of a two-point column vector $\bar{\gamma}$, and we further define a matrix $\mathbf{F}$, such that:

$$
\mathbf{R}=4\left[\begin{array}{c}
R\left(k_{1}\right)  \tag{40}\\
R\left(k_{2}\right) \\
R\left(k_{3}\right) \\
\vdots \\
R\left(k_{N}\right)
\end{array}\right], \quad \bar{\gamma}=\left[\begin{array}{c}
\alpha_{1} \\
\beta_{1}
\end{array}\right], \quad \text { and } \quad \mathbf{F}=\left[\begin{array}{cc}
1 & -2 F\left(k_{1}\right) \\
1 & -2 F\left(k_{2}\right) \\
1 & -2 F\left(k_{3}\right) \\
\vdots & \vdots \\
1 & -2 F\left(k_{N}\right)
\end{array}\right]
$$

then the relationship suggested by equation (39) is given by

$$
\begin{equation*}
\mathbf{F} \bar{\gamma}=\mathbf{R} . \tag{41}
\end{equation*}
$$

Clearly then a solution to this problem involves computation of some approximation $\tilde{\gamma}=$ $\tilde{\mathbf{F}}^{-1} \mathbf{R}$; a least-squares approach is the most obvious.

The 1D normal incidence parameter estimation associated with the inversion of equation (39) is numerically illustrated below, firstly to show how well it works, and secondly to show how poorly it works. Following that we will respond to the latter aspect.

### 5.1 Numeric Examples I: Single Interface

We have a linear set of equations, one for each instance of available wavenumbers $k_{1}, k_{2}, \ldots$

$$
\begin{align*}
& \alpha_{1}-2 \beta_{1} F\left(k_{1}\right)=4 R\left(k_{1}\right), \\
& \alpha_{1}-2 \beta_{1} F\left(k_{2}\right)=4 R\left(k_{2}\right), \\
& \alpha_{1}-2 \beta_{1} F\left(k_{3}\right)=4 R\left(k_{3}\right), \tag{42}
\end{align*}
$$

In fact, estimating $\alpha_{1}$ and $\beta_{1}$ is precisely equivalent to estimating (respectively) the yintercept and slope of a set of data along the axes $4 R(k)$ and $-2 F(k)$. And similarly to the fitting of a line, provided we have perfect data we only require two input wavenumbers to get an answer. Letting these be $k_{1}=\omega_{1} / c_{0}$ and $k_{2}=\omega_{2} / c_{0}$, we may solve for estimates of $\alpha_{1}\left(k_{1}, k_{2}\right)$ and $\beta_{1}\left(k_{1}, k_{2}\right)$ for any pair of $k_{1} \neq k_{2}$ :

$$
\begin{align*}
& \beta_{1}\left(k_{1}, k_{2}\right)=2 \frac{R\left(k_{2}\right)-R\left(k_{1}\right)}{F\left(k_{1}\right)-F\left(k_{2}\right)} \\
& \alpha_{1}\left(k_{1}, k_{2}\right)=4 \frac{R\left(k_{2}\right) F\left(k_{1}\right)-R\left(k_{1}\right) F\left(k_{2}\right)}{F\left(k_{1}\right)-F\left(k_{2}\right)} . \tag{43}
\end{align*}
$$

Equation (43) may be used along with a chosen Earth model to numerically test the efficacy of this inversion. Table 1 contains the details of four models:

| Model | Reference $c_{0}(\mathrm{~m} / \mathrm{s})$ | Non-reference $c_{1}(\mathrm{~m} / \mathrm{s})$ | Non-reference $Q_{1}$ |
| :---: | :--- | :--- | :--- |
| 1 | 1500 | 1800 | 100 |
| 2 | 1500 | 1800 | 10 |
| 3 | 1500 | 2500 | 100 |
| 4 | 1500 | 2500 | 10 |

Table 1: Test models used for the single interface $c, Q$ linear inversion.

Figures $3-6$ show sets of recovered parameters using the respective models in Table 1. For the sake of illustration, frequency pairs $k_{1}=k_{2}$, for which the inversion equations are singular, are smoothed using averages of adjacent $\left(k_{1} \neq k_{2}\right)$ results. The recovered $Q$ values from the the measured viscoacoustic wave field are in error on the order of $\% 1$; this is true for all realistic contrasts in $Q$ (i.e., up to $Q=10$ as tested here). As the wavespeed contrast increases, the recovered $Q$ is in greater error, but even in the large contrast cases of Models 3 and 4 , the error is under $\% 10$. In all cases the error increases at low frequency; it is particularly acute when both $k_{1}$ and $k_{2} \rightarrow 0$. The viscous linear inverse problem involves reflection coefficients that vary with frequency, in other words the "contrast" of the model is also effectively frequency-dependent. In a linear inversion, a frequency-dependent contrast implies a frequency-dependent accuracy level. It is encouraging to see that elsewhere, i.e. at larger $k_{1}, k_{2}$, the nominal acoustic (non-attenuating) Born approximation for the wavespeed is attained. Compare the results of Figure 3 (Model 1), for instance, with the 1D acoustic Born approximation associated with a wavespeed contrast of $1500 \mathrm{~m} / \mathrm{s}$ to $1800 \mathrm{~m} / \mathrm{s}$ (in which $R_{1} \approx 0.091$ ):

$$
\begin{equation*}
c_{1} \approx \frac{c_{0}}{\left(1-\alpha_{1}\right)^{1 / 2}} \mathrm{~m} / \mathrm{s}=\frac{c_{0}}{\left(1-4 R_{1}\right)^{1 / 2}} \mathrm{~m} / \mathrm{s} \approx 1880.3 \mathrm{~m} / \mathrm{s} \tag{44}
\end{equation*}
$$

Since the wavespeed inversion results are very similar to those of a linear Born inversion in the absence of a viscous component, and the $Q$ estimates are within a few percent of the correct value even at the highest reasonable contrast, we may declare this linear inversion example a success.

It has been noted elsewhere that linear Born inversion results tend to worsen in the presence of an unknown overburden, because of unaccounted-for transmission effects. Qualitatively, we might expect the absorptive/dispersive case to suffer greatly from this problem because of the exaggerated transmission effects of the lossy medium on the wave field amplitudes. In other words, if we add a second interface to the model, inversion error can be expected to worsen. Let us next gauge the extent of this error.


Figure 3: Recovered parameters from the normal incidence linearized Born inversion for $Q$ and wavespeed. Left: $Q$ recovery using two (complex) reflection coefficients over a range of frequencies, $k_{1}$ and $k_{2}(\mathrm{~Hz})$. Right: wavespeed recovery utilizing same (complex) reflection coefficients. Model parameters correspond to Model 1 in Table 1.


Figure 4: Recovered parameters from the normal incidence linearized Born inversion for $Q$ and wavespeed. Left: Q recovery using two (complex) reflection coefficients over a range of frequencies, $k_{1}$ and $k_{2}$ (Hz). Right: wavespeed recovery utilizing same (complex) reflection coefficients. Model parameters correspond to Model 2 in Table 1.

### 5.2 Numeric Examples II: Interval $Q$ Estimation

In a single parameter normal incidence problem, i.e. in which acoustic wavespeed contrasts are linearly inverted for from the data by trace integration, profiles may be generated, not just a single interface contrast. A similar procedure may be developed for the 1D normal incidence two parameter problem $(c(z)$ and $Q(z))$. We use the following model to describe


Figure 5: Recovered parameters from the normal incidence linearized Born inversion for $Q$ and wavespeed. Left: $Q$ recovery using two (complex) reflection coefficients over a range of frequencies, $k_{1}$ and $k_{2}(H z)$. Right: wavespeed recovery utilizing same (complex) reflection coefficients. Model parameters correspond to Model 3 in Table 1.


Figure 6: Recovered parameters from the normal incidence linearized Born inversion for $Q$ and wavespeed. Left: $Q$ recovery using two (complex) reflection coefficients over a range of frequencies, $k_{1}$ and $k_{2}(H z)$. Right: wavespeed recovery utilizing same (complex) reflection coefficients. Model parameters correspond to Model 4 in Table 1.
the 1D normal incidence data associated with a model with $N$ interfaces, at each of which the wavespeed and $Q$ values are assumed to alter. The data are

$$
\begin{equation*}
D(k)=\sum_{n=1}^{N} D_{n}(k), \tag{45}
\end{equation*}
$$

such that

$$
\begin{align*}
& D_{n}(k)=R_{n}^{\prime}(k) \exp \left\{\sum_{j=1}^{n} i 2 k_{(j-1)}\left(z_{j}^{\prime}-z_{j-1}^{\prime}\right)\right\}, \\
& R_{n}^{\prime}(k)=R_{n}(k) \prod_{j=1}^{n-1}\left[1-R_{j}^{2}(k)\right],  \tag{46}\\
& k_{(j)}=\frac{\omega}{c_{j}}\left[1+\frac{i}{2 Q_{j}}-\frac{1}{\pi Q_{j}}\left(\frac{k}{k_{r}}\right)\right],
\end{align*}
$$

where, as ever, $k$ is the acoustic reference wavenumber $\omega / c_{0}$, and where $R_{n}(k)$ is the reflection coefficient of the $n$ 'th interface. The variables $z_{j}^{\prime}$ are the true depths of the interfaces. The exponential functions imply an arrival time and a "shape" for each event. Figure 7 shows an example data set of the form of equation (47) for a two-interface case in the conjugate (pseudo-depth) domain, i.e., in which

$$
\begin{align*}
D(k) & =D_{1}(k)+D_{2}(k) \\
& =R_{1}(k) e^{i 2 k z_{1}^{\prime}}+R_{2}^{\prime}(k) e^{i 2 k z_{1}} e^{i 2 k_{(1)}\left(z_{2}^{\prime}-z_{1}^{\prime}\right)} . \tag{47}
\end{align*}
$$

Interpreting the data in terms of the acoustic reference wavespeed $c_{0}$, and with no knowledge of $Q(z)$, equation (47) becomes

$$
\begin{equation*}
D(k)=R_{1}(k) e^{i 2 k z_{1}}+\tilde{R}_{2}(k) e^{i 2 k z_{1}} e^{i 2 k\left(z_{2}-z_{1}\right)} \tag{48}
\end{equation*}
$$

where $z_{j}$ are pseudo-depths. Comparing equations (47) and (48), clearly the apparent reflection coefficient $\tilde{R}_{2}(k)$ has a lot to account for in the absorptive/dispersive case - not just the transmission coefficients $\left(1-R_{1}^{2}(k)\right)$, but now the attenuation as well. This is where the linear approximation is expected to encounter difficulty.

To pose the new interval $Q$ problem, the total perturbations are written as the sum of the perturbations associated with each of these two events:


Figure 7: Example data set of the type used to validate/demonstrate the linearized c, $Q$ profile inversion. (a) Full synthetic trace, consisting of two events, $D_{1}+D_{2}$, plotted in the conjugate (pseudo-depth) domain. The first event corresponds to the contrast from acoustic reference medium to a viscoacoustic layer, and the second corresponds to a deeper viscoacoustic contrast; (b) first event $D_{1}$ (plotted in the conjugate domain); (c) second event $D_{2}$.

$$
\begin{align*}
& \alpha_{1}(-2 k)=\alpha_{11}(-2 k)+\alpha_{12}(-2 k) \\
& \beta_{1}(-2 k)=\beta_{11}(-2 k)+\beta_{12}(-2 k), \tag{49}
\end{align*}
$$

in which $\alpha_{1 n}, \beta_{1 n}$ are the perturbations associated with the event $D_{n}(k)$. Given data $D(k)$ similar to that of Figure 7, then, the linear data equations become

$$
\begin{equation*}
\alpha_{11}(-2 k)+\alpha_{12}(-2 k)-2 F(k)\left[\beta_{11}(-2 k)+\beta_{12}(-2 k)\right]=4 \frac{D(k)}{i 2 k} . \tag{50}
\end{equation*}
$$

Using the assumption of step-like interfaces again, and placing these interfaces at pseudodepth (i.e. imaging with $c_{0}$ ), we make the substitutions

$$
\begin{align*}
& \alpha_{11}(-2 k)=\alpha_{11} \frac{e^{i 2 k z_{1}}}{i 2 k} \\
& \beta_{11}(-2 k)=\beta_{11} \frac{e^{i 2 k z_{1}}}{i 2 k} \\
& \alpha_{12}(-2 k)=\alpha_{12} \frac{e^{i 2 k z_{1}} e^{i 2 k\left(z_{2}-z_{1}\right)}}{i 2 k}  \tag{51}\\
& \beta_{12}(-2 k)=\beta_{12} \frac{e^{i 2 k z_{1}} e^{i 2 k\left(z_{2}-z_{1}\right)}}{i 2 k}
\end{align*}
$$

Then equations (48) and (50) combine to become

$$
\begin{equation*}
\alpha_{11}-2 \beta_{11} F(k)+e^{i 2 k\left(z_{2}-z_{1}\right)}\left[\alpha_{12}-2 \beta_{11} F(k)\right]=4 R_{1}(k)+4 \tilde{R}_{2}(k) e^{i 2 k\left(z_{2}-z_{1}\right)} . \tag{52}
\end{equation*}
$$

From here we may proceed in two different ways. First, we may re-write this relationship as

$$
\begin{equation*}
\alpha_{11}+L_{1}(k) \beta_{11}+L_{2}(k) \alpha_{12}+L_{3}(k) \beta_{11}=\hat{R}(k), \tag{53}
\end{equation*}
$$

where

$$
\begin{align*}
& L_{1}(k)=-2 F(k) \\
& L_{2}(k)=e^{i 2 k\left(z_{2}-z_{1}\right)},  \tag{54}\\
& L_{3}(k)=L_{1}(k) L_{2}(k), \\
& \hat{R}(k)=4 R_{1}(k)+4 \tilde{R}_{2}(k) e^{i 2 k\left(z_{2}-z_{1}\right)},
\end{align*}
$$

and recognize that this constitutes an independent system of linear equations (since $L_{n}(k)$ are known and - usually - differ as $k$ differs) which is overdetermined given greater than four input wavenumbers $k$. This procedure generalizes immediately to $>2$ interfaces, with the caveats (a) larger numbers of input frequencies are required for larger numbers of interfaces, and (b) if events are close to one another such that $z_{n+1}-z_{n} \approx 0$, the procedure becomes less well-posed.

Secondly, if we can gain access to the local frequency content of each reflected event, i.e. if we can individually estimate $R_{1}(k), \tilde{R}_{2}(k)$, etc., then, equating like pseudo-depths in equation (52), we have

$$
\begin{align*}
& \alpha_{11}-2 \beta_{11} F(k)=4 R_{1}(k), \\
& \alpha_{12}-2 \beta_{12} F(k)=4 \tilde{R}_{2}(k) . \tag{55}
\end{align*}
$$

This procedure also immediately generalizes to multiple interfaces, and is therefore a highly specific (in the sense of experimental configuration) approach to "interval $Q$ estimation". It
does not require large numbers of available frequencies as input, regardless of the number of interfaces, but it does require the adequate estimation of $\tilde{R}_{2}(k)$ and any subsequent deeper event $\tilde{R}_{n}(k)$.

We now proceed to demonstrate the latter interval $Q$ estimation approach for two simple models (because of passing interest in this exact problem, and abiding interest in more general cases, e.g. two-parameter with offset, which we expect to behave similarly). Table 2 details the parameters used.

| Model \# | Layer 1 c (m/s) | Layer 1 Q | Layer 2 $\mathrm{c}(\mathrm{m} / \mathrm{s})$ | Layer 2 Q |
| :---: | :--- | :--- | :--- | :--- |
| 1 | 1550 | 200 | 1600 | 10 |
| 2 | 1550 | 100 | 1600 | 10 |

Table 2: Test models used for the single layer $c$, $Q$ linear inversion. The reference medium, $z<500 \mathrm{~m}$, is acoustic and characterized by $c_{0}=1500 \mathrm{~m} / \mathrm{s}$.

Figures 8 - 9 illustrate the inversion for interval $c / Q$ inversion on input data from models $1-2$ using low-valued pairs of input frequencies. Figures $10-11$ illustrate the inversion of the same two models, this time using two high-frequency input reflection coefficients.

Observing the progression of Figures $8-9$, in which the layer $Q$ becomes smaller (and attenuation increases), it is clear that the effective transmission effects of the viscoacoustic medium cause increasing error in the inversion for the lower medium. This is a natural part of linear viscoacoustic inversion, and stands as an indication that the raw estimate of absorptive/dispersive $V_{1}$ has limited value. However, comparing the same inversions using different input frequencies (i.e. Figures $10-11$ compared to Figures $8-9$ ), we also see that the inversion accuracy is dependent on which frequencies are utilized. At low frequency the effects of a viscous overburden negatively affect the inversion results, but the error is much smaller than at high.

To summarize, we may pose the 1D normal incidence two-parameter problem such that (assuming we have access to the reflection coefficients - with transmission error - $\widetilde{R}_{n}(k)$ ) a two-interface case may be handled. This means we may again take advantage of the simplicity of this surrogate version of the two-parameter with offset problem. We straightforwardly illustrate the increased transmission error associated with viscous propagation, but point out that the error is strongly dependent on which frequencies are used as input. Let us next take a closer look at this issue.


Figure 8: Linear c, Q profile inversion for a single layer model (Model 1 in Table 2). (a) Data used in inversion plotted against pseudo-depth $z=c_{0} t / 2$. (b) Recovered wavespeed perturbation $\alpha_{1}(z)$ (solid) plotted against true perturbation (dotted). (c) Recovered $Q$ perturbation $\beta_{1}(z)$ (solid) against true perturbation (dotted). Low input frequencies used.


Figure 9: Linear c, Q profile inversion for a single layer model (Model 2 in Table 2). (a) Data used in inversion plotted against pseudo-depth $z=c_{0} t / 2$. (b) Recovered wavespeed perturbation $\alpha_{1}(z)$ (solid) plotted against true perturbation (dotted). (c) Recovered $Q$ perturbation $\beta_{1}(z)$ (solid) against true perturbation (dotted). Low input frequencies used.


Figure 10: Linear c, Q profile inversion for a single layer model (Model 1 in Table 2). (a) Data used in inversion plotted against pseudo-depth $z=c_{0} t / 2$. (b) Recovered wavespeed perturbation $\alpha_{1}(z)$ (solid) plotted against true perturbation (dotted). (c) Recovered $Q$ perturbation $\beta_{1}(z)$ (solid) against true perturbation (dotted). High input frequencies used.


Figure 11: Linear c, Q profile inversion for a single layer model (Model 2 in Table 2). (a) Data used in inversion plotted against pseudo-depth $z=c_{0} t / 2$. (b) Recovered wavespeed perturbation $\alpha_{1}(z)$ (solid) plotted against true perturbation (dotted). (c) Recovered $Q$ perturbation $\beta_{1}(z)$ (solid) against true perturbation (dotted). High input frequencies used.

### 5.3 The Relationship Between Accuracy and Frequency

The examples of the previous section highlight an inherent source of inaccuracy in the linear inverse output of the absorptive/dispersive problem, namely that the attenuated reflection coefficients lead to parameter estimates that are often greatly in error. In the following section we will consider courses of action we may take to address this problem; here we will simply observe more closely one aspect of the nature of the viscoacoustic linear inverse problem, that may suggest strategies for minimizing the effect of attenuation on the linear result.

We have considered the "1D normal incidence + structural assumptions" problem as a simple surrogate for the "1D + offset" problem. In doing so we are able to cast a two-parameter estimation procedure as an overdetermined system of linear equations to be solved, with one equation/two unknowns for every frequency in the experiment. In the interval $Q$ problem, having chosen two frequencies (and perfect data), we find that as soon as the lower event is significantly attenuated (i.e. the layer $Q$ is strong enough) the $Q$ estimate for the lower medium is deflected far from the true value. However, the deflection is not uniform for input frequency pairs. In this section we more closely consider the input frequencies used.

We apply the procedures of the two event interval $Q$ problem to the sequence of input models/data sets described in Table 3. In this case we consider the output as a function of all possible input frequency pairs, which, similarly to the single-interface case, are plotted as surfaces against these pairs $k_{1}, k_{2}$. See Figures $12-16$.

Observing the evolution of $Q$ estimates for the upper and lower interfaces, a similar but slightly more complete picture is formed. Clearly as $Q_{1}$ becomes lower, and so the input reflection coefficient from the lower interface becomes more and more attenuated, the $Q$ estimates become worse and worse. It is interesting to note, however, that the deflection of $Q_{2}\left(k_{1}, k_{2}\right)$ away from the true $Q_{2}$ is not uniform across frequency/wavenumber pairs. Rather, there is a tendency for the error to increase with higher frequencies. This is an intuitive result, since by its nature the attenuative medium saps the wave field (and therefore the effective reflection coefficient) of energy preferentially at the high frequencies.

It may eventually be profitable to include such insight into the choice of weighting scheme that will be part of the solution of the overdetermined systems, weighting more heavily the contributions from lower frequency pairs. In the analogous "1D with offset" problem this will amount to a judicious weighting of angle, or offset, contributions in the estimation, rather than explicitly frequency contributions.

| Model \# | Layer $1 \mathrm{c}(\mathrm{m} / \mathrm{s})$ | Layer 1 Q | Layer $2 \mathrm{c}(\mathrm{m} / \mathrm{s})$ | Layer 2 Q |
| :---: | :--- | :--- | :--- | :--- |
| 1 | 1550 | 300 | 1600 | 10 |
| 2 | 1550 | 250 | 1600 | 10 |
| 3 | 1550 | 200 | 1600 | 10 |
| 4 | 1550 | 150 | 1600 | 10 |
| 5 | 1550 | 100 | 1600 | 10 |

Table 3: Test models used for the single layer $c, Q$ linear inversion. The reference medium, $z<500 \mathrm{~m}$, is acoustic and characterized by $c_{0}=1500 \mathrm{~m} / \mathrm{s}$.


Figure 12: Linear c, $Q$ profile inversion for a single layer model (Model 1 in Table 3). Top left: $Q$ estimate for top interface; top right: wavespeed estimate for top interface; bottom left: $Q$ estimate for lower interface; bottom right: wavespeed estimate for lower interface. Frequencies (denoted $k_{1}$ and $k_{2}$ ) are in units of Hz .


Figure 13: Linear c, $Q$ profile inversion for a single layer model (Model 2 in Table 3). Top left: $Q$ estimate for top interface; top right: wavespeed estimate for top interface; bottom left: $Q$ estimate for lower interface; bottom right: wavespeed estimate for lower interface. Frequencies (denoted $k_{1}$ and $k_{2}$ ) are in units of Hz .


Figure 14: Linear c, $Q$ profile inversion for a single layer model (Model 3 in Table 3). Top left: $Q$ estimate for top interface; top right: wavespeed estimate for top interface; bottom left: $Q$ estimate for lower interface; bottom right: wavespeed estimate for lower interface. Frequencies (denoted $k_{1}$ and $k_{2}$ ) are in units of Hz .


Figure 15: Linear c, $Q$ profile inversion for a single layer model (Model 4 in Table 3). Top left: $Q$ estimate for top interface; top right: wavespeed estimate for top interface; bottom left: $Q$ estimate for lower interface; bottom right: wavespeed estimate for lower interface. Frequencies (denoted $k_{1}$ and $k_{2}$ ) are in units of Hz .


Figure 16: Linear c, $Q$ profile inversion for a single layer model (Model 5 in Table 3). Top left: $Q$ estimate for top interface; top right: wavespeed estimate for top interface; bottom left: $Q$ estimate for lower interface; bottom right: wavespeed estimate for lower interface. Frequencies (denoted $k_{1}$ and $k_{2}$ ) are in units of Hz .

### 5.4 A Layer-stripping Correction to Linear $Q$ Estimation

There are two proactive ways we could attempt to rectify the problem of decay of reflectivity: (1) correct the linear result with an ad hoc patch, or (2) resort to nonlinear methodologies, since viscoacoustic propagation is a nonlinear effect of the medium parameters on the wave field (Innanen, 2003; Innanen and Weglein, 2003).

The latter approach is material for a subsequent report. For now, we illustrate a correction to the linear inverse results that amounts to a layer stripping strategy. Using the localreflectivity approach of the previous section, recall that we solved for amplitudes of steplike contrasts $\alpha_{1 n}, \beta_{1 n}$ via effective reflection coefficients (equation (51)). In general, the equations are

$$
\begin{align*}
& \alpha_{11}-2 \beta_{11} F(k)=4 R_{1}(k), \\
& \alpha_{12}-2 \beta_{12} F(k)=4 \tilde{R}_{2}(k), \\
& \alpha_{13}-2 \beta_{13} F(k)=4 \tilde{R}_{3}(k),  \tag{56}\\
& \vdots \\
& \alpha_{1 N}-2 \beta_{1 N} F(k)=4 \tilde{R}_{N}(k),
\end{align*}
$$

where

$$
\begin{equation*}
\tilde{R}_{N}(k)=R_{N}^{\prime}(k) \exp \left\{\sum_{j=1}^{N} i 2\left[\frac{\omega}{c_{j-1}} \frac{F(k)}{Q_{j-1}}\right]\left(z_{j}^{\prime}-z_{j-1}^{\prime}\right)\right\} . \tag{57}
\end{equation*}
$$

In other words, the effective reflection coefficients in the data are the desired reflection coefficients (still in error by transmission from the overburden) operated on by an absorption/dispersion factor.

Implementing the low-contrast approximation $z_{j}^{\prime}-z_{j-1}^{\prime} \approx z_{j}-z_{j-1}$, and recognizing that the first step is to estimate $c_{1}$ and $Q_{1}$ from the unaffected reflection coefficient $R_{1}(k)$, we can apply a corrective operator to the next lowest reflection coefficient:

$$
\begin{equation*}
R_{2}^{\prime}(k) \approx \frac{\tilde{R}_{2}(k)}{\exp \left[i 2 \frac{\omega}{c_{1}} \frac{F(k)}{Q_{1}}\left(z_{2}-z_{1}\right)\right]} \tag{58}
\end{equation*}
$$

This approximation is closer to that which equations (56) "would like to see", a reflection coefficient with attenuation corrected-for. This may be done for deeper events also, making use of equation (57) to design appropriate corrective operators.

Figures $17-18$ demonstrate the use of this layer stripping, or "bootstrap", approach to patching up the linearized interval $Q$ estimation procedure for mid range input frequencies; the models used are detailed in Table 2. Clearly the results are far superior, and there
is reason to be encouraged by this approach. A note of caution: the estimated $Q$ values are in error by a small amount due to the linear approximation, so each correction of the next deeper reflection coefficient will be in increasing error. $Q$-compensation of this kind is sensitive to input $Q$ values, so one might expect the approach to eventually succumb to cumulative error.


Figure 17: Linear c, Q profile inversion for Model 1 in Table 2 with attenuative propagation effects compensated for. (a) Data used in inversion plotted against pseudo-depth $z=c_{0} t / 2$. (b) Recovered wavespeed perturbation $\alpha_{1}(z)$ (solid) plotted against true perturbation (dotted). (c) Recovered $Q$ perturbation $\beta_{1}(z)$ (solid) against true perturbation (dotted).


Figure 18: Linear c, Q profile inversion for Model 2 in Table 2 with attenuative propagation effects compensated for. (a) Data used in inversion plotted against pseudo-depth $z=c_{0} t / 2$. (b) Recovered wavespeed perturbation $\alpha_{1}(z)$ (solid) plotted against true perturbation (dotted). (c) Recovered $Q$ perturbation $\beta_{1}(z)$ (solid) against true perturbation (dotted).

## 6 Conclusions

We have demonstrated some elements of a linear Born inversion for wavespeed and $Q$ with arbitrary variation in depth. The development generates a well-posed (overdetermined) estimation scheme for arbitrary distributions two parameters in depth, given shot recordlike data. The nature of this viscoacoustic inversion is such that a 1 D normal incidence version of the problem can be made tractable (and in many ways comparable to the general problem) for two parameters with the assumption of a basic structural form for the model. The simplicity of this casting of the problem makes it useful as a way to develop the basics of the linear viscoacoustic inversion problem.

For a single interface, accuracy is high up to very large $Q$ contrast, with the caveat that the required input to this inversion are the rather subtle spectral properties of the absorptive/dispersive reflection coefficient. For interval $c / Q$ estimation, the attenuation of the reflected events (a process that is nonlinear in the parameters) produces exaggerated trans-
mission error in the estimation; we show that this is mitigated by both judicious choice (or weighting) of input frequencies and/or an ad hoc bootstrap/layer-stripping type correction of lower reflection coefficients.

We develop this linear estimation procedure for two reasons - first in an attempt to produce useful linear $Q$ estimates, and second because this estimate (or something very like it) is the main ingredient for a more sophisticated nonlinear inverse scattering series procedure. Results are encouraging on both fronts.

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# Nonlinear inversion of absorptive/dispersive wave field measurements: preliminary development and tests 

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#### Abstract

We consider elements of the nonlinear inversion of primaries with a particular emphasis on the viscoacoustic, or absorptive/dispersive case. Since the main ingredient to higher-order terms in the inverse scattering series (as it is currently cast) is the linear, or Born, inverse, we begin by considering its "natural" form, i.e. without the corrections and/or assumptions discussed in Innanen and Weglein (2004). The absorptive/dispersive linear inverse for a single interface is found to be a filtered step-function, and for a single layer a set of increasingly smoothed step-functions. The "filters" characterizing the model shape are analyzed.

We next consider the nature of the inversion subseries, which would accept the linear inverse output and via a series of nonlinear operations transform it into a signal with the correct amplitudes (i.e. the true $Q$ profile). The nature of the data event communication espoused by these nonlinear operations is discussed using a simple 1D normal incidence physical milieu, and then we focus on the viscous (attenuating) case. We show that the inversion subseries correctly produces the $Q$ profile if the input signal has had the attenuation (or viscous propagation effects) compensated. This supports the existing ansatz (Innanen and Weglein, 2003) regarding the nature of a generalized imaging subseries (i.e. not the inversion subseries) for the viscous case, as carrying out the task of $Q$ compensation.


## 1 Introduction

The linear viscoacoustic inversion procedure of Innanen and Weglein (2004), including the "pure" linear inversion and the subsequent "patches" used to correct for transmission error, may well be of practical use. The authors show that, numerically, the linear $Q$ estimate is close to the true value in cases where strong nonlinear effects (such as propagation in the non-reference medium) are not present. However in any real situation the nonlinear nature of the relationship between the data and the parameters of interest will be present. Patching the procedure, through the use of intermediate medium parameter estimates, to correct for nonlinearity has been shown to be effective in numerical tests. However, this relies on linear parameter and operator estimates, and hence error will accumulate; furthermore, the application is ad hoc, and in any real medium the analogous procedure would require medium assumptions.

The linear inverse output is important on its own (especially when constructed using judiciously chosen - low - frequency inputs), but also as the potential input to higher order terms in the series. In either case, the quality and fidelity of the linear inversion results are of great importance; what defines those qualities, however, may differ. For instance, consider the imaging subseries of the inverse scattering series (Weglein et al., 2002; Shaw et al., 2003). Its task is to take the linearized inversion, which consists of contrasts that (i) are incorrectly located, and (ii) are of the incorrect amplitude (both in comparison to the true model), and correct the locations of the contrasts without altering the amplitudes. This task is accomplished, in 1D normal incidence examples, by weighting the derivatives of the linearized inversion result by factors which rely on the incorrect (Born) amplitudes. In other words, the imaging subseries relies on the correct provision of the incorrect model values. It doesn't matter that the linear inverse is not close to the true inverse - the inverse scattering series expects this. What matters is that the incorrect linear results are of high fidelity, so that a sophisticated nonlinear inversion procedure may function properly to correct them.
The full inverse scattering series provides a means to carry out viscoacoustic/viscoelastic inversion, in the absence of assumptions about the structure of the medium, and without ad hoc and theoretically inconsistent corrections and patches. This paper addresses some preliminary issues regarding the posing of the nonlinear inverse problem for absorptive/dispersive media, including a discussion and demonstration of the "natural" form for the linear inverse result - important because the linear inverse result is the input for all higher order terms in the various subseries as they are currently cast. We also consider some general aspects of the inversion subseries from the standpoint of the nonlinear, corrective communication between events that is implied by the computation of the terms in this subseries. We finally show results of the inversion subseries performing a nonlinear $Q$-estimation, but using a $Q$ compensated data set, and comment on the implications for nonlinear absorptive/dispersive inversion.

## 2 Input Issues: What Should a Viscous $V_{1}$ Look Like?

Second- and higher-order terms in the inverse scattering series will work to get around the problems of the linear inverse in a theoretically consistent way, one that does not require the linearly solved-for model to be close to the true model. We need to understand what these nonlinear methods expect as input. The subseries use structure and amplitude information in the Born approximation to construct the desired subsurface model; what then of the step-like structural assumptions we made previously? They are unlikely to be appropriate. What we need is to postulate the "natural" form of the linear viscoacoustic inverse, the form expected by the higher order terms in both subseries.

The "patch-up" structural form for the linear inversion of the 1D normal incidence problem was adopted to permit two parameters to be solved-for. We can avoid this assumption and investigate the natural linear inverse form by considering a two parameter-with-offset inversion, as discussed in Innanen and Weglein (2004), or a linear single parameter inversion
from data associated with contrasts in $Q$ only. For the sake of parsimony, we consider the latter here.

### 2.1 Viscous $V_{1}$ for a Single-Interface

Consider first the single interface configuration illustrated in Figure 1. The data reflected from such a scatterer is


Figure 1: Single interface experiment involving a contrast in $Q$ only.

$$
\begin{equation*}
D(k)=R_{Q}(k) e^{i 2 k z_{1}} \tag{1}
\end{equation*}
$$

where as ever, $k=\omega / c_{0}, z_{1}$ is the depth of the $Q$ contrast, and the source and receiver are coincident at $z_{s}=z_{g}=0$. Using terminology developed in Innanen and Weglein (2004), the reflection coefficient is

$$
\begin{equation*}
R_{Q}(k)=-\frac{F(k)}{Q_{1}\left[2+\frac{F(k)}{Q_{1}}\right]} . \tag{2}
\end{equation*}
$$

An example data set is illustrated in Figure 2. General 1D normal incidence data equations for wavespeed perturbations $\left(\alpha_{1}\right)$ and $Q$ perturbations $\left(\beta_{1}\right)$ were produced by Innanen and Weglein (2004):

$$
\begin{equation*}
\alpha_{1}(-2 k)-2 \beta_{1}(-2 k) F(k)=4 \frac{D(k)}{i 2 k} \tag{3}
\end{equation*}
$$

which can be altered to serve our current purposes by setting $\alpha_{1}(-2 k) \equiv 0$. With this alteration we have a single equation and a single unknown for each wavenumber $k$, and thus a well-posed if not overdetermined problem. The revised equations are

$$
\begin{equation*}
\beta_{1}(-2 k)=-2 \frac{D(k)}{i 2 k F(k)} . \tag{4}
\end{equation*}
$$

We can gain some insight by inserting the analytic form for data from a single interface case (equation 1):

$$
\begin{equation*}
\beta_{1}(-2 k)=-2 \frac{R_{Q}(k) e^{i 2 k z_{1}}}{i 2 k F(k)}=-2 \frac{R_{Q}(k)}{F(k)} \frac{e^{i 2 k z_{1}}}{i 2 k} . \tag{5}
\end{equation*}
$$

Recognizing the frequency-domain form for a Heaviside function, and making use of the convolution theorem, the linear form for $\beta_{1}(z)$ in the pseudo-depth $z$ domain is

$$
\begin{equation*}
\beta_{1}(z)=L_{1}(z) * H\left(z-z_{1}\right), \tag{6}
\end{equation*}
$$

where $*$ denotes convolution, and $L_{1}(z)$ is a spatial filter of the form

$$
\begin{equation*}
L_{1}(z)=\int_{-\infty}^{\infty} e^{-i 2 k z} L_{1}(-2 k) d(-2 k) \tag{7}
\end{equation*}
$$

in which

$$
\begin{equation*}
L_{1}(-2 k)=\frac{2 R_{Q}(k)}{F(k)}=\frac{2}{2 Q_{1}+\left[\frac{i}{2}-\frac{1}{\pi} \ln \left(\frac{k}{k_{r}}\right)\right]} \tag{8}
\end{equation*}
$$

the latter arising from the substitution of the reflection coefficient. From the point of view of equation (6), the linear inverse output will be a Heaviside function altered (filtered) by a function that depends on the reflection coefficient and $F(k)$; obviously, the more this filter $L_{1}(z)$ differs from a delta function, the more the linear absorptive/dispersive reconstruction will be perturbed away from the (correct) step-like form. Figures $3-5$ illustrate this combination and the resulting form for $\beta_{1}(z)$ for several values of $Q_{1}$.


Figure 2: Example numerical data for a single interface experiment with acoustic reference/viscoacoustic non-reference contrast.

At large $Q$ contrast (i.e. Figure 5) there is a distinct effect on the spatial distribution of the recovered linear perturbation. Nevertheless, compared to the result that would be produced by simply integrating the data $D(k)$ (in which there is a strong filtering $R_{Q}(k)$ upon what would usually be a delta-like event) the effect is small. In other words, there has been an attempt by the linear inverse formalism to correct for the filtering effects of $R_{Q}(k)$. By
inspection of equation (4), the correction takes the form of spectral division of the data $D(k)$ by $F(k)$.


Figure 3: Recovered single parameter viscoacoustic model, $Q_{1}=100$ : (a) Heaviside component with step at $z_{1}$, (b) Filter $L_{1}(z)$ to be convolved with Heaviside component, (c) $\beta_{1}(z)=L_{1}(z) * H\left(z-z_{1}\right)$. At this contrast level, $L_{1}(z)$ is close to a delta-function and there is little effect other than a scaling of the Heaviside function.


Figure 4: Recovered single parameter viscoacoustic model, $Q_{1}=50$ : (a) Heaviside component with step at $z_{1}$, (b) Filter $L_{1}(z)$ to be convolved with Heaviside component, (c) $\beta_{1}(z)=L_{1}(z) * H\left(z-z_{1}\right)$. At this contrast level, there is a slight visible effect on the Heaviside, including a slight "droop" with depth.


Figure 5: Recovered single parameter viscoacoustic model, $Q_{1}=10$ : (a) Heaviside component with step at $z_{1}$, (b) Filter $L_{1}(z)$ to be convolved with Heaviside component, (c) $\beta_{1}(z)=L_{1}(z) * H\left(z-z_{1}\right)$. At this contrast level, $L_{1}(z)$ produces a noticeable effect on the spatial structure of the recovered parameter distribution.

### 2.2 Viscous $V_{1}$ for Multiple-Interfaces

The linear inversion for a single-interface $\beta_{1}(z)$ produces a filtered Heaviside function whose spectrum is an imperfect correction of the first integral of the viscoacoustic reflection coefficient. Numerical examples indicate that the correction is good up to high $Q$ contrast, after which a distinct impact on the recovered spatial distribution of $\beta_{1}(z)$ is noticed. Any deeper reflectors will produce data events which are likewise determined by absorptive/dispersive reflectivity, but more importantly will have been affected by viscous propagation.


Figure 6: Single layer experiment involving contrasts in $Q$ only.
Consider the single viscoacoustic layer example of Figure 6, which produces data qualitatively like the example in Figure 7. The data from such an experiment has the form

$$
\begin{equation*}
D(k)=R_{Q 1}(k) e^{i 2 k z_{1}}+R_{Q 2}^{\prime}(k) e^{i 2 k z_{1}} e^{i 2 k_{1}\left(z_{2}-z_{1}\right)} \tag{9}
\end{equation*}
$$

where $k_{1}$ is the propagation wavenumber within the layer, generally for this example

$$
\begin{equation*}
k_{j}=\frac{\omega}{c_{0}}\left[1+\frac{F(k)}{Q_{j}}\right], \tag{10}
\end{equation*}
$$

(notice that the wavespeed remains $c_{0}$ ), and the reflection coefficient is

$$
\begin{equation*}
R_{Q 2}^{\prime}(k)=\left[1-R_{Q 1}(k)\right]^{2} \frac{k_{1}-k_{2}}{k_{1}+k_{2}}=\left[1-R_{Q 1}(k)\right]^{2} \frac{F(k)\left[\frac{1}{Q_{1}}-\frac{1}{Q_{2}}\right]}{2+F(k)\left[\frac{1}{Q_{1}}+\frac{1}{Q_{2}}\right]} . \tag{11}
\end{equation*}
$$

By separating the "ballistic" component of $k_{1}$ from the absorptive/dispersive component, the data may be re-written as

$$
\begin{equation*}
D(k)=\left[R_{Q 1}(k)\right] e^{i 2 k z_{1}}+\left[R_{Q 2}^{\prime}(k) e^{i 2 k \frac{F(k)}{Q_{1}}\left(z_{2}-z_{1}\right)}\right] e^{i 2 k z_{2}} . \tag{12}
\end{equation*}
$$

We can again gain some insight into the form of $\beta_{1}(z)$ by substituting in the analytic data in equation (12):

$$
\begin{align*}
\beta_{1}(-2 k) & =-2 \frac{D(k)}{i 2 k F(k)} \\
& =\left[-\frac{2 R_{1 Q}(k)}{F(k)}\right] \frac{e^{i 2 k z_{1}}}{i 2 k}+\left[-\frac{2 R_{Q 2}^{\prime}(k) e^{i 2 k \frac{F(k)}{Q_{1}}\left(z_{2}-z_{1}\right)}}{F(k)}\right] \frac{e^{i 2 k z_{2}}}{i 2 k}  \tag{13}\\
& =L_{1}(-2 k) \frac{e^{i 2 k z_{1}}}{i 2 k}+L_{2}(-2 k) \frac{e^{i 2 k z_{2}}}{i 2 k}
\end{align*}
$$

where

$$
\begin{align*}
& L_{1}(-2 k)=-\frac{2 R_{1 Q}(k)}{F(k)} \\
& L_{2}(-2 k)=-\frac{2 R_{Q 2}^{\prime}(k) e^{i 2 k \frac{F(k)}{Q_{1}}\left(z_{2}-z_{1}\right)}}{F(k)} \tag{14}
\end{align*}
$$

are once again filters whose inverse Fourier transforms $L_{1}(z)$ and $L_{2}(z)$ act upon Heaviside functions:

$$
\begin{equation*}
\beta_{1}(z)=L_{1}(z) * H\left(z-z_{1}\right)+L_{2}(z) * H\left(z-z_{2}\right) . \tag{15}
\end{equation*}
$$

The second filter $L_{2}$ contains components that will impart effects of attenuative propagation onto the linear inverse result - see equation (14). Meanwhile, the only "processing" that occurs is the deconvolution of $F(k)$, which we have seen goes some way towards correcting for the phase/amplitude effects of the viscoacoustic reflectivity. The linear inversion will therefore include the unmitigated effects of propagation in its output.

Figures 8 - 9 illustrate the linear reconstruction of the two-event, single layer problem. Not surprisingly there is a characteristic smoothness - and error - in the linear inverse results below the first interface.

In summary, we may utilize a one-parameter, 1D normal incidence absorptive/dispersive linear inverse milieu to investigate the "natural" spatial form of the viscoacoustic Born approximation. This is useful in that (1) such a form is what the higher order terms of


Figure 7: Example numerical data for a single layer experiment with acoustic reference/viscoacoustic nonreference contrast.
the series will be expecting - also, it is the normal-incidence version of the form we may expect from the multi-parameter, 1D-with offset inversion output developed theoretically by Innanen and Weglein (2004) - and (2) we see what remains to be done by the higher-order terms in the series.

The results are consistent with the statements of Innanen (2003) and Innanen and Weglein (2003); namely, that amplitude adjustment or [nonlinear] $Q$-estimation is required, and that depropagation, or nonlinear $Q$-compensation (in which the smooth edges of Figures 8 - 9 are sharpened) is also required. The balance of this paper is concerned with the former task; the latter is considered future work.

## 3 Nonlinear Event Communication in Inversion

At the same time that we consider specifically viscoacoustic inputs to the nonlinear inversion procedures afforded us by the inverse scattering series, we further consider some general aspects of the inversion subseries - i.e., those component terms of the series which devote themselves to the task of parameter identification. In particular, we comment in this section on the nature of the communication between events that occurs during the computation of the inversion subseries, and what that implies regarding the relative importance of nonlinear combinations of events.

Consider momentarily the relationship between a simple 1D normal incidence acoustic data set and its corresponding Born inverse. Imagine the data set to be made up of the reflected primaries associated with a single layer of wavespeed $c_{1}$, bounded from above by a half space of wavespeed $c_{0}$, and below by a half space of wavespeed $c_{2}$. If the interfaces are at $z_{1}>0$ and $z_{2}>z_{1}$, and the source and receiver are at $z_{s}=z_{g}=0$, the data is

$$
\begin{equation*}
D(k)=R_{1} e^{i 2 \frac{\omega}{c_{0}} z_{1}}+R_{2}^{\prime} e^{i 2 \frac{\omega}{c_{0}} z_{1}} e^{i 2 \frac{\omega}{c_{1}}\left(z_{2}-z_{1}\right)} \tag{16}
\end{equation*}
$$

where $R_{2}^{\prime}=\left(1-R_{1}\right)^{2} R_{2}$ contains the effects of shallower transmission effects, and $R_{1}$ and $R_{2}$ are the reflection coefficients of the upper and lower interfaces respectively. If $c_{0}$ is taken


Figure 8: Illustration of a recovered single parameter viscoacoustic model, $Q_{1}=100, Q_{2}=20$ : (a) Filter $L_{1}(z)$ to be convolved with Heaviside component $H\left(z-z_{1}\right)$, (b) Filter $L_{2}(z)$ to be convolved with Heaviside component $H\left(z-z_{2}\right)$, (c) $\beta_{1}(z)=L_{1}(z) * H\left(z-z_{1}\right)+L_{2}(z) * H\left(z-z_{2}\right)$. The propagation effects on the estimate of the lower interface are apparent.
to be the wavespeed everywhere, and pseudo-depths $z_{1}^{\prime}$ and $z_{2}^{\prime}$ are assigned to events of half the measured traveltime and this reference wavespeed assumption, equation (16) may be written

$$
\begin{equation*}
D(k)=R_{1} e^{i k z_{1}^{\prime}}+R_{2}^{\prime} e^{i k z_{2}^{\prime}} \tag{17}
\end{equation*}
$$

where $k=\omega / c_{0}$. We now drop the primes in $z$ for convenience, but remain in the pseudodepth domain for the duration of this section. In the space domain this reads

$$
\begin{equation*}
D(z)=R_{1} \delta\left(z-z_{1}\right)+R_{2}^{\prime} \delta\left(z-z_{2}\right) . \tag{18}
\end{equation*}
$$

Using standard terminology, the Born inverse associated with the above data set, $\alpha_{1}(z)=$ $V_{1}(k, z) / k^{2}$, is given by

$$
\begin{equation*}
\alpha_{1}(z)=4 \int_{0}^{z} D\left(z^{\prime \prime}\right) d z^{\prime \prime}=4 R_{1} H\left(z-z_{1}\right)+4 R_{2}^{\prime} H\left(z-z_{2}\right) \tag{19}
\end{equation*}
$$



Figure 9: Illustration of a recovered single parameter viscoacoustic model, $Q_{1}=20, Q_{2}=10$ : (a) Filter $L_{1}(z)$ to be convolved with Heaviside component $H\left(z-z_{1}\right)$, (b) Filter $L_{2}(z)$ to be convolved with Heaviside component $H\left(z-z_{2}\right)$, (c) $\beta_{1}(z)=L_{1}(z) * H\left(z-z_{1}\right)+L_{2}(z) * H\left(z-z_{2}\right)$. The propagation effects on the estimate of the lower interface are increasingly destructive, and produce a characteristic smoothing.

The inversion subseries then acts upon this quantity nonlinearly:

$$
\begin{equation*}
\alpha_{I N V}(z)=\sum_{j=1}^{\infty} A_{j} \alpha_{1}^{j}(z) \tag{20}
\end{equation*}
$$

(where $A_{j}=\left[j(-1)^{j-1}\right] /\left[4^{j-1}\right]$ ) i.e. by summing weighted powers of $\alpha_{1}(z)$.
Let us begin by considering the form of the Born inverse result of equation (19). The Heaviside sum is in some sense not a good way of expressing the local amplitude of $\alpha_{1}(z)$. Another way of writing it is

$$
\begin{equation*}
\alpha_{1}(z)=4 R_{1}\left[H\left(z-z_{1}\right)-H\left(z-z_{2}\right)\right]+\left(4 R_{1}+4 R_{2}^{\prime}\right) H\left(z-z_{2}\right), \tag{21}
\end{equation*}
$$

which now expressly gives the local amplitudes. This is somewhat more edifying in the sense that it explicitly makes reference to the linear (additive) communication between events
implied by the linear inversion for the lower layer (i.e. $R_{1}$ and $R_{2}^{\prime}$ ). Let us make this even more explicit with a definition. Let:

$$
\begin{equation*}
\alpha_{1}(z)=D_{1}\left[H\left(z-z_{1}\right)-H\left(z-z_{2}\right)\right]+\left(D_{1}+D_{2}\right) H\left(z-z_{2}^{\prime}\right), \tag{22}
\end{equation*}
$$

where $D_{1}=4 R_{1}, D_{2}=4 R_{2}^{\prime}$, or more generally $D_{n}=4 R_{n}^{\prime}$, the subscript referring explicitly to the event in the data that contributed to that portion of the inverse. Equation (22) straightforwardly generalizes to $N$ layers, if we define $H\left(z-z_{N+1}\right) \equiv 0$ :

$$
\begin{equation*}
\alpha_{1}(z)=\sum_{n=1}^{N}\left(\sum_{i=1}^{n} D_{i}\right)\left[H\left(z-z_{n}\right)-H\left(z-z_{n+1}\right] .\right. \tag{23}
\end{equation*}
$$

Figure 10 illustrates these definitions for the two-layer case. The Heavisides disallow interaction between the terms in brackets (.) (for different $n$ values) under exponentiation, so

$$
\begin{equation*}
\alpha_{1}^{j}(z)=\sum_{n=1}^{N}\left(\sum_{i=1}^{n} D_{i}\right)^{j}\left[H\left(z-z_{n}\right)-H\left(z-z_{n+1}\right)\right] . \tag{24}
\end{equation*}
$$

This allows us the flexibility to consider the local behaviour of the inversion result $\alpha_{I N V}(z)$ directly in terms of the data events $D_{k}$ that contribute to it. The amplitude of the $j$ 'th term in the inversion subseries for the correction of the $m$ 'th layer is

$$
\begin{equation*}
A_{j}\left(\sum_{i=1}^{m} D_{i}\right)^{j} \tag{25}
\end{equation*}
$$

For the second layer, this amounts to

$$
\begin{equation*}
A_{j}\left(D_{1}+D_{2}\right)^{j} \tag{26}
\end{equation*}
$$

and for the third layer,

$$
\begin{equation*}
A_{j}\left(D_{1}+D_{2}+D_{3}\right)^{j} \tag{27}
\end{equation*}
$$

Notice that in terms of $D_{n}$, the computation of the inversion subseries will produce terms similar to the exponentiation of a multinomial. Therefore each term of the inversion subseries (in this simple 1D normal incidence case) will involve a set of subterms whose coefficients are given by the multinomial formula. For instance, the first three terms for the second layer are

$$
\begin{align*}
& A_{1}\left(D_{1}+D_{2}\right) \\
& A_{2}\left(D_{1}^{2}+2 D_{1} D_{2}+D_{2}^{2}\right)  \tag{28}\\
& A_{3}\left(D_{1}^{3}+3 D_{1}^{2} D_{2}+3 D_{1} D_{2}^{2}+D_{2}^{3}\right)
\end{align*}
$$

etc., i.e. they depend strongly on the familiar coefficients of the binomial formula:

$$
\begin{align*}
& 1 \\
& 11 \\
& 121 \\
& 1331  \tag{29}\\
& 14641
\end{align*}
$$

Meanwhile the third layer follows trinomial-like terms, and so forth. This has an interesting consequence. Consider the third-order term of the third layer:

$$
\begin{equation*}
A_{3}\left(D_{1}^{3}+D_{2}^{3}+D_{3}^{3}+3 D_{1}^{2} D_{2}+3 D_{1}^{2} D_{3}+3 D_{2}^{2} D_{1}+3 D_{2}^{2} D_{3}+3 D_{3}^{2} D_{1}+3 D_{3}^{2} D_{2}+6 D_{1} D_{2} D_{3}\right) . \tag{30}
\end{equation*}
$$

In equation (30): all else being equal (i.e. if we assume that events $D_{1}, D_{2}$, and $D_{3}$ are of approximately equal size), the term which combines all events evenly, $D_{1} D_{2} D_{3}$, can be seen to be six times as important as a term which involves a single event only, e.g. $D_{1}^{3}$. The multinomial formula therefore not only informs us about the distribution of communicating data events produced by the inversion subseries, but it also predicts the relative importance of one kind of communication over another. If we believe what we see in equation (30), nonlinear interaction terms which use information as evenly-distributed as possible from all data events above the layer of interest will dominate.

Since the binomial/multinomial formulas are used to model the relative probabilities of the outcomes of random experiments, the inter-event communication distribution problem is analogous to any number of well-known betting games, such as throwing dice or tossing coins. For instance, the relative probabilities of observing heads ( $h$ ) vs. tails $(t)$ in $n$ coin tosses is given by the binomial formula:

$$
\begin{equation*}
(t+h)^{n}=\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} t^{n-j} h^{j}, \tag{31}
\end{equation*}
$$

and the rolling of dice is similarly given by the multinomial formula for a 6 -term generating function.

The inversion subseries is of course anything but random or stochastic. The point of the analogy is to show that, especially at high order, the contribution of evenly distributed nonlinear data communication is greater than that of any other type. For the two event case, the relative importance of even-distribution over biased distribution is precisely the same as the relative probability of tossing equal numbers of heads and tails over large numbers of either. In other words, the nonlinear inversion terms which simultaneously make use of as
many data events as are available dwarfs any other contribution. This behaviour may be speaking of some intuitively reasonable feature of the inversion subseries, that in inverting for the amplitude of a deep layer, the global effect of what overlies it is far more important than any one (or few) local effect(s).


Figure 10: Synthetic data (a) corresponding to a two-layer model; (b) the Born approximation $\alpha_{1}(z)$. Local increases or decreases in the linear inversion due to specific data events are labelled $D_{n}$.

## 4 Nonlinear $Q$-estimation

In this section some properties of the inversion subseries as it addresses the reconstruction of attenuation contrasts from 1D seismic data are considered. A peculiar data set is created for this purpose, one that is designed not to reflect reality, but rather to examine certain aspects of the inverse scattering series and only those aspects. Also, it involves the choice of a simplified model of attenuation. We begin with a description of the physical framework of this analysis.

We use the following two dispersion relations:

$$
\begin{align*}
& k=\frac{\omega}{c_{0}}, \\
& k(z)=\frac{\omega}{c_{0}}[1+i \beta(z)], \tag{32}
\end{align*}
$$

the first being that of the reference medium, and the second that of a non reference medium in which the wavespeed may vary, as may the parameter $\beta$, which, like the wavespeed
is considered an inherent property of the medium in which the wave propagates. It is responsible for not only altered reflection and transmission properties of a wave, but also for the amplitude decay associated with intrinsic friction. One could argue pretty reasonably that $\beta$ is related to the better known parameter $Q$ by $\beta=1 / 2 Q$ because of their similar roles in changing the amplitude spectrum of an impulse. However, most $Q$ models are more sophisticated in that they utilize dispersion to ensure the causality of the medium response. We make use of this "friction" model, which captures much of the key behaviour of attenuating media, to take advantage of its simplicity.

For this investigation, we restrict attention to models which contain variations in $\beta$ only. Waves propagate with wavespeed $c_{0}$ everywhere.

### 4.1 A Further Complex Reflection Coefficient

Contrasts in $\beta$ only will produce reflections with strength

$$
\begin{equation*}
R_{n}=\frac{k_{n-1}-k_{n}}{k_{n-1}+k_{n}}=\frac{i\left(\beta_{n-1}-\beta_{n}\right)}{2+i\left(\beta_{n-1}+\beta_{n}\right)}, \tag{33}
\end{equation*}
$$

and, since we assume an acoustic reference medium ( $\beta_{0}=0$ ),

$$
\begin{equation*}
R_{1}=\frac{-i \beta_{1}}{2+i \beta_{1}} \tag{34}
\end{equation*}
$$

As in the dispersive models used elsewhere in this paper, the reflection coefficient is complex. Without dispersion, the complex reflection coefficient implies the combination of a frequency independent phase rotation and a real reflection coefficient: $R_{1}=\left|R_{1}\right| e^{i \theta}$. This is akin to the reflection coefficients discussed in Born and Wolf (1999).

### 4.2 A Non-attenuating Viscoacoustic Data Set

A further simplification has to do with the measured data, rather than the medium itself. As in the acoustic case, true viscoacoustic data will be a series of transient pulses that return, delayed by an appropriate traveltime and weighted by a combination of reflection and (shallower) transmission coefficients. But unlike an acoustic medium, the propagation between contrasts also determines the character of the events. Each pulse that has travelled through a medium with $\beta \neq 0$ will have had its amplitude suppressed at a cycle-independent rate, and therefore be broadened. Although this is a fundamental aspect of viscoacoustic wave propagation (and indeed one of the more interesting aspects for data processing), for the purposes of this investigation we suppress it in creating the data. The results to follow will help explain why.

The perturbation $\gamma$ is based on an attenuative propagation law, and is itself complex:

$$
\begin{equation*}
\gamma(z)=1-\frac{k^{2}(z)}{k_{0}^{2}}=-2 i \beta(z)+\beta^{2}(z) \tag{35}
\end{equation*}
$$

If the propagation effects are suppressed (or have been corrected for already), then the Born approximation of $\gamma(z)$, namely $\gamma_{1}(z)$ is

$$
\begin{equation*}
\gamma_{1}(z)=A_{1} H\left(z-z_{1}\right)+A_{2} H\left(z-z_{2}\right)+\ldots \tag{36}
\end{equation*}
$$

where the $A_{n}$ are given by

$$
\begin{equation*}
A_{n}=4 R_{n} \prod_{j=1}^{n-1}\left(1-R_{j}^{2}\right) \tag{37}
\end{equation*}
$$

and the reflection coefficients $R_{j}(k)$ are given by equations (33) and (34).
In the previous section it was noted that the contributing terms to any layer in the inversion subseries can be found by invoking the multinomial formula; and the contributing terms to a two-layer model are produced by the binomial formula (which has a much more simply computable form). We assume a two-layer model, in which the reference medium ( $c=c_{0}$, $\beta=0$ ) is perturbed below $z_{1}$ such that $c=c_{0}$ and $\beta=\beta_{1}$, and again below $z_{2}$ such that $c=c_{0}$ and $\beta=\beta_{2}$. The depths $z_{1}$ and $z_{2}$ are chosen to be 300 m and 600 m respectively. This results in

$$
\begin{equation*}
\gamma_{1}(z)=\left[\frac{-4 i \beta_{1}}{2+i \beta_{1}}\right] H\left(z-z_{1}\right)+\left[\frac{-4 i\left(\beta_{1}-\beta_{2}\right)}{2+i\left(\beta_{1}+\beta_{2}\right)}\left(1-\frac{\beta_{1}^{2}}{\left(2+i \beta_{1}\right)^{2}}\right)\right] H\left(z-z_{2}\right) \tag{38}
\end{equation*}
$$

The $n$ 'th power of the Born approximation for either layer of interest is computable by appealing to the binomial formula. The result of the inversion subseries on this two interface model is therefore

$$
\begin{equation*}
\gamma_{I N V}(z)=A_{1}^{I N V} H\left(z-z_{1}\right)+A_{2}^{I N V} H\left(z-z_{2}\right) \tag{39}
\end{equation*}
$$

where

$$
\begin{align*}
& A_{1}^{I N V}=\sum_{j=1}^{\infty} \frac{j(-1)^{j-1}}{4^{j-1}} A_{1}^{j} \\
& A_{2}^{I N V}=\sum_{j=1}^{\infty} \frac{j(-1)^{j-1}}{4^{j-1}}\left(\sum_{k=1}^{j} \frac{j!}{k!(j-k)!} A_{1}^{j-k} A_{2}^{k}\right) \tag{40}
\end{align*}
$$

and where

$$
\begin{align*}
& A_{1}=\frac{-4 i \beta_{1}}{2+i \beta_{1}} \\
& A_{2}=\frac{-4 i\left(\beta_{1}-\beta_{2}\right)}{2+i\left(\beta_{1}+\beta_{2}\right)}\left(1-\frac{\beta_{1}^{2}}{\left(2+i \beta_{1}\right)^{2}}\right) \tag{41}
\end{align*}
$$

are the amplitudes of the Born approximation in equation (36). Figures 11 - 13 demonstrate numerically the convergence of these expressions (the real part, for the sake of illustration) towards the true value. There is clear oscillation over the first three or so terms for the second layer, but the oscillations settle down to close to the correct value with reasonable speed. This suggests an important aspect of going beyond the Born approximation in $Q$ estimation - the deeper layers, whose amplitudes are related not only to their own $Q$ value but to those above them, require the invocation of several nonlinear orders to achieve their correct value.

Finally, Figures 14 - 16 illustrate the same inversion results as those seen in Figures 11 13, but in the context of the spatial distribution of the reconstructed perturbation.


Figure 11: Convergence of the viscoacoustic inversion subseries for a two-interface model, consisting of contrasts in (friction-model) attenuation parameter. The real component of the perturbation is plotted against inversion order or iteration number (connected dots) and with respect to the true perturbation value (solid).
(a) layer 1 inversion. (b) layer 2 inversion. For model contrasts $\beta_{1}=0.15, \beta_{2}=0.25, \gamma_{I N V}(z)$ is close to $\gamma(z)$ for both layers.


Figure 12: Convergence of the viscoacoustic inversion subseries for a two-interface model, consisting of contrasts in (friction-model) attenuation parameter. The real component of the perturbation is plotted against inversion order or iteration number (connected dots) and with respect to the true perturbation value (solid). (a) layer 1 inversion. (b) layer 2 inversion. For model contrasts $\beta_{1}=0.35, \beta_{2}=0.55, \gamma_{I N V}(z)$ differs slightly from $\gamma(z)$ in the lower layer.


Figure 13: Convergence of the viscoacoustic inversion subseries for a two-interface model, consisting of contrasts in (friction-model) attenuation parameter. The real component of the perturbation is plotted against inversion order or iteration number (connected dots) and with respect to the true perturbation value (solid). (a) layer 1 inversion. (b) layer 2 inversion. For model contrasts $\beta_{1}=0.55, \beta_{2}=0.35, \gamma_{I N V}(z)$ again differs slightly from $\gamma(z)$ in the lower layer.


Figure 14: Convergence of the viscoacoustic inversion subseries for a two- interface model, consisting of contrasts in (friction-model) attenuation parameter. Depth profile of the real component of the constructed perturbation (solid) is plotted against the true perturbation (dotted) for 5 iterations. (a) iteration 1 (Born approximation) - (e) iteration 5. Model inputs: $\beta_{1}=0.15, \beta_{2}=0.25$.


Figure 15: Convergence of the viscoacoustic inversion subseries for a two- interface model, consisting of contrasts in (friction-model) attenuation parameter. Depth profile of the real component of the constructed perturbation (solid) is plotted against the true perturbation (dotted) for 5 iterations. (a) iteration 1 (Born approximation) - (e) iteration 5. Model inputs: $\beta_{1}=0.35, \beta_{2}=0.55$.


Figure 16: Convergence of the viscoacoustic inversion subseries for a two- interface model, consisting of contrasts in (friction-model) attenuation parameter. Depth profile of the real component of the constructed perturbation (solid) is plotted against the true perturbation (dotted) for 5 iterations. (a) iteration 1 (Born approximation) - (e) iteration 5. Model inputs: $\beta_{1}=0.55, \beta_{2}=0.35$.

The success of the inversion subseries in reconstructing the amplitudes of the attenuation contrasts in the examples of the previous section is worth pursuing, since the success was achieved by using a strange input data set; one in which the propagation effects of the medium had been stripped off a priori. How should this success be interpreted?

It follows that if the inversion subseries correctly uses the above data set to estimate $Q$ - as opposed to one in which the propagation effects were present - then the subseries must have been expecting to encounter something of that nature.

The apparent conclusion is that the other non-inversion subseries terms should be looked-to to provide just such an attenuation-compensated Born approximation. This is in agreement with the conclusions reached by Innanen and Weglein (2003), in which it appeared that the terms representing the viscous analogy to the imaging subseries would be involved in the more general task of removing the effects of propagation (reflector location and attenuation/dispersion) on the wave field.

More generally, these results suggest that for these simple cases at least (1D normal incidence acoustic/viscoacoustic) we see distinct evidence that the inversion subseries produces corrected parameter amplitudes when the output of the imaging/depropagation subseries is used as input. This is a strong departure from the natural behavior of the inverse scattering series, which, of course, does not use output of one set of terms as input for another.

## 5 Conclusions

The aims of this paper have been to explore (1) the nature of the input to the nonlinear absorptive/dispersive methods implied by the mathematics of the inverse scattering series, and (2) the nature of the inversion subseries, or target identification subseries itself, both generally and with respect to the viscoacoustic problem.

Regarding the input, the viscoacoustic Born approximation is shown to have two characteristic elements - an amplitude/phase effect due to the reflectivity, and an absorptive/dispersive effect due to the propagation effects; the former is ameliorated somewhat in the linear inverse procedure, and the former is not. Hence the linear $Q$-profile is reconstructed with some account taken of the phase and amplitude spectra of the local events, but without any attempt to correct for the effective transmission associated with viscous propagation.

The nonlinear processing and inversion ideas fleshed out in this paper rely, in their detail, on specific models of absorption and dispersion. For instance, much of the formalism in the Born inversion for $c$ and $Q$ in the single interface case arises because the frequency dependence of the dispersion law $(F(k))$ is separable, multiplying a frequency-independent $Q$. Since the detail of the inversion relies on the good fortune of having a viscoacoustic model that behaves so accommodatingly, it is natural to ask: if we have tied ourselves to one particular model of dispersion, do we not then rise and fall with the (case by case) utility and accuracy of this one model? The practical answer is yes, of course; the specific formulae
and methods in this paper require that this constant $Q$ model explain the propagation and reflection/transmission in the medium. But one of the core ideas of this paper has been to root out the basic reasons why such an approach might work. The dispersion-based frequency dependence of the reflection coefficient as a means to separate $Q$ from $c$ is not a model dependent idea, regardless of model-to-model differences in detail and difficulty.

The more sophisticated $Q$ model employed in this paper provided a complex, frequencydependent scattering potential, which has since been used to approximate the $Q$ profile in linear inversion, and to comment on the expected spatial form of the input to higher order terms. However, at an early stage in the derivation the form was simplified with a rejection of terms quadratic in $Q$. One might notice that in the latter portion of this paper the same step was not taken in the friction model perturbation that was used as input in the higher order terms. It is anticipated that the more sophisticated quadratic potential will be needed in the dispersive $Q$ model when it is used as series input.

The results of this paper appear to confirm the predictions of the forward series analysis in Innanen and Weglein (2003); i.e. that prior to the inversion (target identification) step, the subseries terms which are nominally concerned with imaging, must also remove the effects of attenuative propagation. The confirmation in this paper is circumstantial but compelling: the ubiquitous increase in quality of the inverse results (both low order constant $Q$ and high order friction-based attenuation) in the absence of propagation effects strongly suggest that depropagated ( $Q$ compensated) inputs are expected in these inversion procedures. A clear future direction for research is in generalizing the imaging subseries to perform $Q$ compensation simultaneously with reflector location.

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# Investigating the grouping of inverse scattering series terms: simultaneous imaging and inversion I 

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#### Abstract

We consider those portions of the inverse scattering series (for the simple case of a 1D normal incidence acoustic problem) that are concerned with imaging and inversion of primaries in a measured wave field. We show that many of the terms involved in this effort can be captured with a subseries that involves the $n$ 'th derivative of the $n$ 'th power of the first integral of the Born approximation; it is referred to here as the simultaneous imaging and inversion subseries. The value of this subseries at present lies in what it can tell us about the functioning of the series as a whole; we investigate it for purposes of basic insight.

We begin by outlining the formulation of the simultaneous imaging and inversion subseries, initially using it to reproduce terms from the 1D normal incidence inverse series as a means to illustrate the extent of the approximations made, and note that an equivalent form for the coupled subseries may be developed which leads to a closed form. We lastly interpret the form of the simultaneous imaging and inversion algorithm from a signal-processing perspective, to provide insight into precisely what operations the inverse scattering series advocates as the general means for imaging and inverting primaries.


## 1 Introduction

The idea of task separation has been the critical conceptual leap in the success of the inverse scattering series to date (Weglein et al., 1997; Weglein et al., 2003). It hinges on the apparent willingness, or even predisposition, of the series to compartmentalize the entire inversion into portions that greatly resemble existing steps in seismic data processing. Task separation is twice beneficial, because ( $i$ ) separately accomplishing any one task disengages the user from the other, often far more complex (and possibly divergent) problems of inversion, and (ii) the output of this task may be a valuable product in its own right. Amongst other advances, a leading-order imaging subseries has been identified (Shaw et al., 2003) in the 1D normal incidence case for which $(i)$ a term of arbitrary order may be immediately written down, and (ii) a closed-form expression exists. This subseries has been shown to locate reflectors with a high-degree of accuracy without concerning itself with the issue of perturbation amplitude (i.e. the 1D version of the inversion task). Furthermore, the second term in the inversion subseries has been shown to improve the estimation of density and bulk modulus (and thereby
$P$-wave velocity etc.) beyond the linearized form for a 1D case with offset. This amounts to a method for nonlinear AVO (Zhang and Weglein, 2003). Hence, early evidence is strongly suggestive of the value of task separation.

We are, however, interested in the functioning of the inverse scattering series as a whole we are interested in exposing the basic mechanisms of the series as a means to transform primaries into correctly-located Earth parameter distributions. That is the core motivation behind the work described in this paper, in which we purposefully couple the tasks of imaging and inversion and investigate the nature of the resulting algorithm. This paper constitutes part of the thesis research of one of us (Innanen, 2003), and has been partly reported in (Innanen and Weglein, 2003), and also includes a recent mathematical treatment.

We begin by outlining the formulation of the simultaneous imaging and inversion approach. We initially use the formulation to reproduce terms from the 1D normal incidence inverse series as a means to illustrate the extent of the approximations made. Then, retracing steps, we detail how, making these approximations, the form of the terms was deduced, and, in particular, how these approximations result in reducing the effect of entire classes of terms in the inverse series to simple alterations of the sign of the output. We then note that an equivalent form for the coupled subseries may be developed which leads to a closed form. Finally we interpret the form of the simultaneous imaging and inversion algorithm, to provide insight into precisely what operations the inverse scattering series advocates as the general means for imaging and inverting primaries. We show how a simultaneous imaging/inversion operator, whose form is dictated by the inverse scattering series, performs imaging via identification and correction of discontinuities in the measured data, and inversion via an alteration of the amplitudes that comes from the exponentiation of the integrals of the measured signal.

## 2 Background and a Useful Notation

This section follows closely the derivation and discussion of Weglein et al. (2003). To briefly review the theory of inverse scattering, it is useful to temporarily resort to an operator notation, whereby, for instance, a "true" wave field satisfies the equation

$$
\begin{equation*}
\mathbf{L G}=-\mathbf{I} \tag{1}
\end{equation*}
$$

and a "reference" wave field satisfies

$$
\begin{equation*}
\mathrm{L}_{0} \mathrm{G}_{0}=-\mathbf{I}, \tag{2}
\end{equation*}
$$

where $\mathbf{L}$ and $\mathbf{L}_{\mathbf{0}}$ are the true and reference wave operators, and $\mathbf{G}$ and $\mathbf{G}_{\mathbf{0}}$ are the true and reference Green's operators respectively. The operators are general in the sense of model, and fully 3D. Equations (1) and (2) are in the space/temporal frequency domain. Two important quantities are associated with the difference of these operators:

$$
\begin{equation*}
\mathbf{V}=\mathbf{L}-\mathbf{L}_{\mathbf{0}} \tag{3}
\end{equation*}
$$

known as the perturbation operator, scattering potential, or scattering operator, and

$$
\begin{equation*}
\Psi_{s}=\mathbf{G}-\mathbf{G}_{\mathbf{0}}, \tag{4}
\end{equation*}
$$

known as the scattered wave field. The Lippmann-Schwinger equation is an operator identity in this framework:

$$
\begin{equation*}
\Psi_{s}=\mathbf{G}-\mathbf{G}_{\mathbf{0}}=\mathbf{G}_{\mathbf{0}} \mathbf{V G} \tag{5}
\end{equation*}
$$

and it begets the Born series through self-substitution:

$$
\begin{align*}
\Psi_{s} & =\mathbf{G}_{\mathbf{0}} \mathbf{V G}_{\mathbf{0}}+\mathbf{G}_{\mathbf{0}} \mathbf{V G}_{\mathbf{0}} \mathbf{V G}_{\mathbf{0}}+\mathbf{G}_{\mathbf{0}} \mathbf{V G}_{\mathbf{0}} \mathbf{V G}_{\mathbf{0}} \mathbf{V G}_{\mathbf{0}}+\ldots  \tag{6}\\
& =\left(\Psi_{s}\right)_{1}+\left(\Psi_{s}\right)_{2}+\left(\Psi_{s}\right)_{3}+\ldots
\end{align*}
$$

In other words, the scattered field is represented as a series in increasing order in the scattering potential. This formalism constitutes a forward modelling of the wave field; it is a nonlinear mapping between the perturbation and the wave field, the latter being written in increasing orders of the former.

Inversion, or the solution for $\mathbf{V}$ from measurements of the scattered field outside of $\mathbf{V}$, has no closed form. The approach taken here is that of Jost and Kohn (1952) and Moses (1956); it was formulated for the inversion of wave velocity by Razavy (1975), and discussed in the framework of seismic data processing and inversion by Stolt and Jacobs (1981) and Weglein et al. (1981). It is to represent the solution (the perturbation operator) as an infinite series:

$$
\begin{equation*}
\mathbf{V}=\mathbf{V}_{1}+\mathbf{V}_{\mathbf{2}}+\mathbf{V}_{\mathbf{3}}+\ldots \tag{7}
\end{equation*}
$$

where $\mathbf{V}_{\mathbf{j}}$ is " $j$ 'th order in the data". This form is substituted into the terms of the Born series, and terms of like order in $\Psi_{s}$ are equated (each term is considered to have been evaluated on the measurement surface $m$ ). This is the form of the inverse scattering series:

$$
\begin{aligned}
& \left(\Psi_{s}\right)_{m}=\left(\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}}\right)_{m}, \\
& 0=\left(\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{2}} \mathbf{G}_{\mathbf{0}}\right)_{m}+\left(\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}}\right)_{m}, \\
& 0=\left(\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{3}} \mathbf{G}_{\mathbf{0}}\right)_{m}+\left(\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}}\right)_{m}+\left(\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{2}} \mathbf{G}_{\mathbf{0}}\right)_{m}+\left(\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{2}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}}\right)_{m},
\end{aligned}
$$

The idea is that $\mathbf{V}_{\mathbf{1}}$, the component of $\mathbf{V}$ that is linear in the data, is solved for with the first equation. This result is substituted into the second equation, leaving $\mathbf{V}_{\mathbf{2}}$ as the only
unknown, which may then also be solved for. This continues until a sufficient set of $\mathbf{V}_{\mathbf{j}}$ are known to accurately approximate the desired result V.

The form of the operators $\mathbf{L}, \mathbf{G}, \mathbf{L}_{\mathbf{0}}, \mathbf{G}_{\mathbf{0}}$, and $\mathbf{V}$ obviously vary depending on the desired form for the wave propagation (i.e. acoustic constant density, elastic, viscoacoustic, etc.), and, in the case of $\mathbf{V}$, are particularly dependent on how propagation in the reference medium differs from that of the true medium. We focus on the simplest possible cases: here that of 1D constant density acoustic media. Reference media are kept homogeneous, and the scattering potential is considered to be confined to a finite region on one side of the source and receiver locations.

This choice amounts to defining

$$
\begin{align*}
& \mathbf{L}=\frac{d^{2}}{d z^{2}}+\left(\frac{\omega}{c(z)}\right)^{2}  \tag{9}\\
& \mathbf{L}_{\mathbf{0}}=\frac{d^{2}}{d z^{2}}+\left(\frac{\omega}{c_{0}}\right)^{2}
\end{align*}
$$

in which case

$$
\begin{equation*}
\mathbf{V}=\left(\frac{\omega}{c(z)}\right)^{2}-\left(\frac{\omega}{c_{0}}\right)^{2}=k^{2} \alpha(z) \tag{10}
\end{equation*}
$$

where $k=\omega / c_{0}$ and $\alpha(z)=1+c_{0}^{2} / c^{2}(z)$. This simple physical framework also permits the use of the Green's function

$$
\begin{equation*}
G_{0}\left(z \mid z_{s} ; k\right)=\frac{e^{i k\left|z-z_{s}\right|}}{2 i k} \tag{11}
\end{equation*}
$$

(which becomes $\mathbf{G}_{\mathbf{0}}$ when it is included as part of the kernel of the integrals of the series). In this framework the inverse scattering series terms of interest (equation (7)) can be reduced to

$$
\begin{equation*}
\alpha(z)=\alpha_{1}(z)+\alpha_{2}(z)+\alpha_{3}(z)+\ldots \tag{12}
\end{equation*}
$$

Finally, following the conventional physical interpretation of the "rightmost" Green's operator in every term of equation (8) as being the incident wave field, these are replaced with incident plane-waves $\psi_{0}\left(z \mid z_{s} ; k\right)$.

The terms of equation (8) become

$$
\begin{align*}
\psi_{s}\left(z \mid z_{s} ; k\right) & =\int_{-\infty}^{\infty} G_{0}\left(z \mid z^{\prime} ; k\right) k^{2} \alpha_{1}\left(z^{\prime}\right) \psi_{0}\left(z^{\prime} \mid z_{s} ; k\right) d z^{\prime} \\
0 & =\int_{-\infty}^{\infty} G_{0}\left(z \mid z^{\prime} ; k\right) k^{2} \alpha_{2}\left(z^{\prime}\right) \psi_{0}\left(z^{\prime} \mid z_{s} ; k\right) d z^{\prime}  \tag{13}\\
& +\int_{-\infty}^{\infty} G_{0}\left(z \mid z^{\prime} ; k\right) k^{2} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} G_{0}\left(z^{\prime} \mid z^{\prime \prime} ; k\right) k^{2} \alpha_{1}\left(z^{\prime \prime}\right) \psi_{0}\left(z^{\prime \prime} \mid z_{s} ; k\right) d z^{\prime \prime} d z^{\prime}
\end{align*}
$$

Consider these equations individually. Solving the first for $\alpha_{1}(z)$ is the 1D equivalent of Born inversion, and amounts, in essence, to trace integration. The resemblance of the equation to a Fourier transform results in (Weglein et al., 2003)

$$
\begin{equation*}
\alpha_{1}(z)=4 \int_{0}^{z} \psi_{s}\left(z^{\prime}\right) d z^{\prime} \tag{14}
\end{equation*}
$$

where the depth variable here is the so-called "pseudo-depth", determined by the natural time variable of the measurement of the wave field $\psi_{s}$ and the reference wavespeed profile.

Weglein et al. (2003) approach the subsequent orders of $\alpha(z)$ by casting them all in terms of the Born approximation $\alpha_{1}(z)$. Also, choices for breaking the integrals up are made based on the resulting form of the terms in orders of $\alpha_{1}$; these forms are by no means the only way to solve the integrals of equation (13); they are a reasoned choice, the basis for the separation of tasks into those of inversion and of imaging.

The formalism described above is based on the assumption that one measures the "scattered field" $\psi_{s}=\psi-\psi_{0}$. It is further assumed (as mentioned above) that all the scatterers occur "beneath" the source and receiver. These assumptions and others lead to a set of requirements on the data that characterize approaches, for the treatment of primary reflections, based on the inverse scattering series:

1. The source waveform has been compensated for.
2. The direct wave has been removed.
3. The source and receiver "ghosts" have been removed.
4. The free-surface multiples have been removed.
5. The internal (interbed) multiples have been removed.

We describe various manipulations of the mathematics of the (previously discussed) casting of the terms of the inverse scattering series. Here we present a notation for these integrals (Innanen, 2003) which speeds up some of the manipulations based on the chain rule and integration by parts. Subsequently, we use this notation to derive the 2 nd order terms of the inverse series, and later, to derive and simplify 3rd and 4th order terms as well.

The terms in the inverse series are often profitably cast as an increasingly complex set of operations on $\alpha_{1}(z)$, the Born approximate solution for $\alpha(z)$, which is linear in the data. Of particular importance is the operation

$$
\begin{equation*}
\int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}\right) d z^{\prime} \tag{15}
\end{equation*}
$$

and "nested" versions of the same, for instance

$$
\begin{equation*}
\int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{z^{\prime}} \alpha_{1}\left(z^{\prime \prime}\right) d z^{\prime \prime} d z^{\prime} \tag{16}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{z^{\prime}} \alpha_{1}\left(z^{\prime \prime}\right) \int_{-\infty}^{z^{\prime \prime}} \alpha_{1}\left(z^{\prime \prime \prime}\right) d z^{\prime \prime \prime} d z^{\prime \prime} d z^{\prime} \tag{17}
\end{equation*}
$$

etc. Notice that equation (15) is a linear operator applied to $\alpha_{1}(z)$, namely the convolution of $\alpha_{1}(z)$ with a "left-opening" Heaviside function. Define this operator, i.e. convolution with a left-opening Heaviside, as $\mathcal{H}\{\cdot\}$, such that

$$
\begin{equation*}
\mathcal{H}\left\{\alpha_{1}(z)\right\}=\int_{-\infty}^{\infty} H\left(z-z^{\prime}\right) \alpha_{1}\left(z^{\prime}\right) d z^{\prime}=\int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}\right) d z^{\prime} \tag{18}
\end{equation*}
$$

Although it will be used sparingly, we further define the convolution of $\alpha_{1}(z)$ with the time reverse of these Heaviside functions, that is, the "right-opening" kind. Let:

$$
\begin{equation*}
\mathcal{H}^{-}\left\{\alpha_{1}(z)\right\}=\int_{-\infty}^{\infty} H\left(z^{\prime}-z\right) \alpha_{1}\left(z^{\prime}\right) d z^{\prime}=\int_{z}^{\infty} \alpha_{1}\left(z^{\prime}\right) d z^{\prime} \tag{19}
\end{equation*}
$$

The following relationship between $\mathcal{H}$ and $\mathcal{H}^{-}$proves useful: for any $f(z)$ and $g(z)$,

$$
\begin{equation*}
\int_{-\infty}^{\infty} f\left(z^{\prime}\right) \mathcal{H}^{-}\left\{g\left(z^{\prime}\right)\right\} d z^{\prime}=\int_{-\infty}^{\infty} g\left(z^{\prime}\right) \mathcal{H}\left\{f\left(z^{\prime}\right)\right\} d z^{\prime} \tag{20}
\end{equation*}
$$

It can be derived by substituting $\mathcal{H}^{-}$into the left-hand side of equation (20) and switching variables. An explicit form of this manipulation is done regularly in the derivation of the terms of the inverse series. We have developed this method/notation simply to speed up the derivations.

Since in this paper the operator $\mathcal{H}\{\cdot\}$ is often (but not exclusively) applied to $\alpha_{1}(z)$, for convenience call

$$
\begin{equation*}
H=\mathcal{H}\left\{\alpha_{1}(z)\right\} . \tag{21}
\end{equation*}
$$

Since the operator $\mathcal{H}$ is essentially the antiderivative operator, it follows that for any $f(z)$ that is confined to a finite region,

$$
\begin{equation*}
\mathcal{H}\left\{\frac{d f(z)}{d z}\right\}=f(z) \tag{22}
\end{equation*}
$$

The nesting seen in equations (16) and (17) is incorporated into this operator framework straightforwardly. Define

$$
\begin{equation*}
\mathcal{H}_{2}\left\{\alpha_{1}(z)\right\}=\mathcal{H}\left\{\alpha_{1}(z) \mathcal{H}\left\{\alpha_{1}(z)\right\}\right\}=\int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{z^{\prime}} \alpha_{1}\left(z^{\prime \prime}\right) d z^{\prime \prime} d z^{\prime} \tag{23}
\end{equation*}
$$

and
$\mathcal{H}_{3}\left\{\alpha_{1}(z)\right\}=\mathcal{H}\left\{\alpha_{1}(z) \mathcal{H}\left\{\alpha_{1}(z) \mathcal{H}\left\{\alpha_{1}(z)\right\}\right\}\right\}=\int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{z^{\prime}} \alpha_{1}\left(z^{\prime \prime}\right) \int_{-\infty}^{z^{\prime \prime}} \alpha_{1}\left(z^{\prime \prime \prime}\right) d z^{\prime \prime \prime} d z^{\prime \prime} d z^{\prime}$,
and so on. In general,

$$
\begin{equation*}
\mathcal{H}_{n}\left\{\alpha_{1}(z)\right\}=\mathcal{H}\left\{\alpha_{1}(z) \mathcal{H}_{n-1}\left\{\alpha_{1}(z)\right\}\right\} \tag{25}
\end{equation*}
$$

This nesting notation, i.e. $\mathcal{H}_{n}\left\{\alpha_{1}(z)\right\}$ will not be retained. Since

$$
\begin{equation*}
\frac{d H}{d z}=\alpha_{1}(z) \tag{26}
\end{equation*}
$$

equation (23) can be written

$$
\begin{equation*}
\mathcal{H}_{2}\left\{\alpha_{1}(z)\right\}=\mathcal{H}\left\{\frac{d H}{d z} H\right\} \tag{27}
\end{equation*}
$$

Also, since

$$
\begin{equation*}
\frac{1}{2} \frac{d H^{2}}{d z}=\frac{d H}{d z} H \tag{28}
\end{equation*}
$$

equation (22) can be used to write

$$
\begin{equation*}
\mathcal{H}_{2}\left\{\alpha_{1}(z)\right\}=\frac{1}{2} H\left\{\frac{d H^{2}}{d z}\right\}=\frac{1}{2} H^{2} . \tag{29}
\end{equation*}
$$

Using this result in the expression for $\mathcal{H}_{3}$, with similar arguments, and continuing on to $\mathcal{H}_{n}$, the general relationship

$$
\begin{equation*}
\mathcal{H}_{n}\left\{\alpha_{1}(z)\right\}=\frac{1}{n!} H^{n} \tag{30}
\end{equation*}
$$

may be derived. The simplification implicit in equation (30) is not trivial - computing the right-hand side with a given $\alpha_{1}$ is much simpler than computing the left.
This terminology is used in the next section, to assist in the derivation and exposition of the terms in the inverse scattering series, and again (and more extensively) in subsequent sections to develop and analyze simultaneous imaging and inversion.
Summing $\alpha_{1}+\alpha_{2}+\ldots$ implies the provision of the model $\alpha$ in terms of the measured wave field and the reference media; hence it is inversion, and imaging, without knowledge of, or determination of, the true wavespeed structure of the medium. Understanding the mechanisms of a series with this promise is made possible by study of the simplest possible models. We reproduce the 2 nd order terms in the inverse series as cast by Weglein et al. (2003), using the operator notation defined above. The simplification achievable by the notation is not noticeable until later sections; here the idea is to show where in the process of deriving terms it becomes applicable.

The second order terms in the inverse series form the equation

$$
\begin{align*}
& \int_{-\infty}^{\infty} G_{0}\left(z \mid z^{\prime} ; k\right) k^{2} \alpha_{2}\left(z^{\prime}\right) \psi_{0}\left(z^{\prime} \mid z_{s} ; k\right) d z^{\prime}=  \tag{31}\\
& \quad-\int_{-\infty}^{\infty} G_{0}\left(z \mid z^{\prime} ; k\right) k^{2} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} G_{0}\left(z^{\prime} \mid z^{\prime \prime} ; k\right) k^{2} \alpha_{1}\left(z^{\prime \prime}\right) \psi_{0}\left(z^{\prime \prime} \mid z_{s} ; k\right) d z^{\prime \prime} d z^{\prime}
\end{align*}
$$

Upon substitution of the forms of the Green's functions etc. into equation (31), many like factors cancel. Furthermore, the left hand side of the equation, like in the Born approximate case, is a Fourier transform of the perturbation component ( $\alpha_{2}$ in this case). Hence equation (31) may be written

$$
\begin{equation*}
\alpha_{2}(-2 k)=-\frac{1}{4}(i 2 k) \int_{-\infty}^{\infty} e^{i k z^{\prime}} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} e^{i k\left|z^{\prime}-z^{\prime \prime}\right|} \alpha_{1}\left(z^{\prime \prime}\right) e^{i k z^{\prime \prime}} d z^{\prime \prime} d z^{\prime} \tag{32}
\end{equation*}
$$

and, treating the two cases demanded by the absolute value bars, one has

$$
\begin{align*}
\alpha_{2}(-2 k)= & -\frac{1}{4}(i 2 k) \int_{-\infty}^{\infty} e^{i k z^{\prime}} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{z} e^{i k\left(z^{\prime}-z^{\prime \prime}\right)} \alpha_{1}\left(z^{\prime \prime}\right) e^{i k z^{\prime \prime}} d z^{\prime \prime} d z^{\prime} \\
& -\frac{1}{4}(i 2 k) \int_{-\infty}^{\infty} e^{i k z^{\prime}} \alpha_{1}\left(z^{\prime}\right) \int_{z}^{\infty} e^{i k\left(z^{\prime \prime}-z^{\prime}\right)} \alpha_{1}\left(z^{\prime \prime}\right) e^{i k z^{\prime \prime}} d z^{\prime \prime} d z^{\prime} \\
= & -\frac{1}{4} \int_{-\infty}^{\infty}(i 2 k) e^{i 2 k z^{\prime}} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} H\left(z^{\prime}-z^{\prime \prime}\right) \alpha_{1}\left(z^{\prime \prime}\right) e^{i k z^{\prime \prime}} d z^{\prime \prime} d z^{\prime}  \tag{33}\\
& -\frac{1}{4} \int_{-\infty}^{\infty}(i 2 k) \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{\infty} H\left(z^{\prime \prime}-z^{\prime}\right) \alpha_{1}\left(z^{\prime \prime}\right) e^{i 2 k z^{\prime \prime}} d z^{\prime \prime} d z^{\prime} .
\end{align*}
$$

The operators defined previously are applicable at this stage, i.e. with the appearance of the convolutions of various quantities with Heaviside functions. Equation (33) can be written

$$
\begin{align*}
\alpha_{2}(-2 k)= & -\frac{1}{4} \int_{-\infty}^{\infty}(i 2 k) e^{i 2 k z^{\prime}}\left[\alpha_{1} H\right]\left(z^{\prime}\right) d z^{\prime} \\
& -\frac{1}{4} \int_{-\infty}^{\infty}(i 2 k)\left[\alpha_{1} \mathcal{H}^{-}\left\{e^{i 2 k z^{\prime}} \alpha_{1}\right\}\right]\left(z^{\prime}\right) d z^{\prime} \\
= & -\frac{1}{4} \int_{-\infty}^{\infty}(i 2 k) e^{i 2 k z^{\prime}}\left[\alpha_{1} H\right]\left(z^{\prime}\right) d z^{\prime}  \tag{34}\\
& -\frac{1}{4} \int_{-\infty}^{\infty}(i 2 k)\left[e^{i 2 k z^{\prime}} \alpha_{1} H\right]\left(z^{\prime}\right) d z^{\prime}
\end{align*}
$$

Because the $\mathcal{H}, \mathcal{H}^{-}$operators are convolution operators, the totality of their action in the integrands in equation (34) is an expression with $z^{\prime}$ dependence only. For convenience we therefore use the variable $z^{\prime}$ freely inside these operators even though strictly speaking it doesn't belong ${ }^{1}$. For instance,

$$
\begin{equation*}
\mathcal{H}^{-}\left\{e^{i 2 k z^{\prime}} \alpha_{1}\right\}=\int_{z^{\prime}}^{\infty} e^{i 2 k z^{\prime \prime}} \alpha_{1}\left(z^{\prime \prime}\right) d z^{\prime \prime} \tag{35}
\end{equation*}
$$

Both terms in equation (34) are Fourier transforms of derivatives with respect to $z$, so

$$
\begin{align*}
\alpha_{2}(z) & =-\frac{1}{4} \frac{d}{d z}\left[2 \alpha_{1} H\right] \\
& =-\frac{1}{2}\left(\left[\frac{d \alpha_{1}}{d z}\right] H+\alpha_{1}^{2}\right) . \tag{36}
\end{align*}
$$

Finally, therefore,

$$
\begin{equation*}
\alpha_{2}(z)=\alpha_{2}^{(1)}(z)+\alpha_{2}^{(2)}(z)=-\frac{1}{2} \alpha_{1}^{2}(z)-\frac{1}{2}\left[\frac{d \alpha_{1}}{d z}\right] \int_{0}^{z} \alpha_{1}\left(z^{\prime}\right) d z^{\prime} . \tag{37}
\end{equation*}
$$

Using similar manipulations, the third order terms may be derived as Weglein et al. (2003):

$$
\begin{equation*}
\alpha_{3}(z)=\alpha_{3}^{(1)}(z)+\alpha_{3}^{(2)}(z)+\alpha_{3}^{(3)}(z)+\alpha_{3}^{(4)}(z)+\alpha_{3}^{(5)}(z) \tag{38}
\end{equation*}
$$

where

[^2]\[

$$
\begin{align*}
& \alpha_{3}^{(1)}(z)=\frac{1}{8}\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right]\left(\int_{0}^{z} \alpha_{1}\left(z^{\prime}\right) d z^{\prime}\right)^{2} \\
& \alpha_{3}^{(2)}(z)=\frac{3}{16} \alpha_{1}^{3}(z), \\
& \alpha_{3}^{(3)}(z)=\frac{3}{4} \alpha_{1}(z)\left[\frac{d \alpha_{1}}{d z}\right] \int_{0}^{z} \alpha_{1}\left(z^{\prime}\right) d z^{\prime},  \tag{39}\\
& \alpha_{3}^{(4)}(z)=-\frac{1}{8}\left[\frac{d \alpha_{1}}{d z}\right] \int_{0}^{z} \alpha_{1}^{2}\left(z^{\prime}\right) d z^{\prime}, \\
& \alpha_{3}^{(5)}(z)=-\frac{1}{16} \int_{0}^{z} \int_{0}^{z}\left[\frac{d \alpha_{1}\left(z^{\prime}\right)}{d z^{\prime}}\right]\left[\frac{d \alpha_{1}\left(z^{\prime \prime}\right)}{d z^{\prime \prime}}\right] \alpha_{1}\left(z^{\prime \prime}+z^{\prime}-z\right) d z^{\prime \prime} d z^{\prime} .
\end{align*}
$$
\]

This specific casting of the inverse series, as in equations (37) and (39), naturally separates the full inversion process into tasks (Weglein et al., 2003). For instance, at each order there is a term which is a weighted power of the Born approximation. Finding these in equations (37) and (39) and summing produces the subseries

$$
\begin{equation*}
\alpha_{I N V}(z)=\alpha_{1}(z)-\frac{1}{2} \alpha_{1}^{2}(z)+\frac{3}{16} \alpha_{1}^{3}(z)+\ldots \tag{40}
\end{equation*}
$$

This subseries has been identified (Weglein et al., 2003; Zhang and Weglein, 2003) and developed as the subseries that is concerned with target identification, or inversion proper.

Similarly, in each order are found terms that involve derivatives of the Born approximation $\alpha_{1}(z)$, weighted by integrals of the same; summing these produces

$$
\begin{equation*}
\alpha_{L O I S I}(z)=\alpha_{1}(z)-\frac{1}{2}\left[\frac{d \alpha_{1}}{d z}\right] \int_{0}^{z} \alpha_{1}\left(z^{\prime}\right) d z^{\prime}+\frac{1}{8}\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right]\left(\int_{0}^{z} \alpha_{1}\left(z^{\prime}\right) d z^{\prime}\right)^{2}+\ldots \tag{41}
\end{equation*}
$$

This subseries (hereafter referred to as LOIS) turns out (Shaw et al., 2003) to be the "leading order imaging subseries", that is, the leading order set of terms which are concerned with the correct location of reflectors in the subsurface. This identification continues; $\alpha_{3}^{(5)}(z)$ in equation (39) is the leading order internal multiple eliminator.

## 3 Simultaneous Imaging and Inversion

We begin this section with the statement of a formula for simultaneous imaging and inversion for this 1D normal incidence acoustic framework. We then back-track somewhat to give an idea of how the expression was derived. The motivation stems ultimately from the fact that in the terms of the inverse scattering series, formulated as they are above, certain combinations of operations on the Born approximation repeatedly arise. It appears that many of these
terms might be produced by a core "generating expression", that for this reason would be equivalent to an engine for the imaging and inversion of the input.

Subsequently this formula is explored regarding (i) its ability to reproduce terms in the inverse series, and (ii) the simplifications inherent in its neglect of a class of series terms.

### 3.1 A Quantity Related to Imaging and Inversion

Consider the quantity

$$
\begin{equation*}
I_{n}(z)=K_{n} \frac{d^{n} H^{n}}{d z^{n}} \tag{42}
\end{equation*}
$$

where

$$
\begin{equation*}
K_{n}=\frac{(-1)^{n-1}}{n}\left(\frac{1}{2^{(n-1)}}\right)^{2}\left[\sum_{k=0}^{n-1} \frac{1}{k!(n-k-1)!}\right] . \tag{43}
\end{equation*}
$$

To compute this quantity, the Born approximation $\alpha_{1}(z)$ is integrated once to get $H=$ $\mathcal{H}\left\{\alpha_{1}(z)\right\}$. The $n$ 'th power is taken, followed by the $n$ 'th derivative; finally it is weighted by $K_{n}$.

This quantity appears to be intimately connected with the terms of the inverse scattering series which relate to imaging and inversion. In and of itself, it is merely an expression that specifies a combination of derivative orders and (effective) numbers of nested integrals of the Born approximation.

### 3.2 Mapping Between $K_{n} d^{n} H^{n} / d z^{n}$ and $\alpha_{n}$

One can best investigate equation (42) by carrying out the $n$ 'th derivative on the $n$ 'th power of $H$, without specifying $\alpha_{1}(z)$, and seeing what happens. In this section equation (42) is expanded in this way for $n=1, n=2, n=3, n=4$ and $n=5$. The results are compared with existing derivations of the $\alpha_{n}(z)$ to clarify which aspects of the inverse problem are addressed by computing them. For convenience we sometimes suppress the $z$ dependence of the Born approximation. At all times $\alpha_{1}$ implies $\alpha_{1}(z)$.

## Expansion for the $n=1$ Term:

If we set $n=1$ then $K_{1}=1$, and by equation (42),

$$
\begin{equation*}
I_{1}(z)=\frac{d H}{d z}=\alpha_{1}(z) \tag{44}
\end{equation*}
$$

i.e. the Born approximation.

## Expansion for the $n=2$ Term:

For the $n=2$ term we have

$$
\begin{equation*}
K_{2}=-\frac{1}{4}\left[\frac{1}{2}+\frac{1}{2}\right]=-\frac{1}{4} . \tag{45}
\end{equation*}
$$

Meanwhile,

$$
\begin{equation*}
\frac{d^{2} H^{2}}{d z^{2}}=\frac{d}{d z}\left[2 H \alpha_{1}\right]=2 \alpha_{1}^{2}+2\left[\frac{d \alpha_{1}}{d z}\right] H \tag{46}
\end{equation*}
$$

Recalling the definition of $H$ in equation (21),

$$
\begin{equation*}
I_{2}(z)=K_{2} \frac{d^{2} H^{2}}{d z^{2}}=-\frac{1}{2} \alpha_{1}^{2}(z)-\frac{1}{2}\left[\frac{d \alpha_{1}}{d z}\right] \int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}\right) d z^{\prime} \tag{47}
\end{equation*}
$$

Comparison of equations (37) and (47) demonstrates that, up to second order, the expression in equation (42) reproduces all of the expected inverse scattering series terms:

$$
\begin{equation*}
I_{2}(z)=\alpha_{2}(z) . \tag{48}
\end{equation*}
$$

## Expansion for the $n=3$ Term:

Proceeding as before, the third term is found by computing

$$
\begin{equation*}
K_{3}=\frac{1}{16}\left[\frac{1}{6}+\frac{1}{3}+\frac{1}{6}\right]=\frac{1}{24}, \tag{49}
\end{equation*}
$$

and then

$$
\begin{align*}
\frac{d^{3} H^{3}}{d z^{3}} & =\frac{d^{2}}{d z^{2}}\left[3 H^{2} \alpha_{1}\right] \\
& =3 \frac{d}{d z}\left[2 H \alpha_{1}^{2}+H^{2}\left[\frac{d \alpha_{1}}{d z}\right]\right] \\
& =3\left[2 \alpha_{1}^{3}+4 \alpha_{1}\left[\frac{d \alpha_{1}}{d z}\right] H+2 \alpha_{1} H\left[\frac{d \alpha_{1}}{d z}\right]+H^{2}\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right]\right]  \tag{50}\\
& =6 \alpha_{1}^{3}+18\left[\frac{d \alpha_{1}}{d z}\right] \alpha_{1} H+3\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right] H^{2} .
\end{align*}
$$

All together,

$$
\begin{equation*}
I_{3}(z)=K_{3} \frac{d^{3} H^{3}}{d z^{3}}=\frac{1}{4} \alpha_{1}^{3}+\frac{3}{4}\left[\frac{d \alpha_{1}}{d z}\right] \alpha_{1} H+\frac{1}{8}\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right] H^{2} \tag{51}
\end{equation*}
$$

or, explicitly,

$$
\begin{equation*}
I_{3}(z)=\frac{1}{4} \alpha_{1}^{3}+\frac{3}{4}\left[\frac{d \alpha_{1}}{d z}\right] \alpha_{1}\left(\int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}\right) d z^{\prime}\right)+\frac{1}{8}\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right]\left(\int_{-\infty}^{z} \alpha_{1}\left(z^{\prime}\right) d z^{\prime}\right)^{2} . \tag{52}
\end{equation*}
$$

These terms no longer match up one-to-one with the full set of inverse scattering series terms; the difference is due to approximations implied by equation (42), which are investigated in the next section. What is missing with this suppression is not merely the leading order internal multiple eliminator, but two other terms as well, including one which alters the coefficient of the inversion $\left(\alpha_{1}^{3}\right)$ term. This can be established by comparing equations (52) and (38).

On the other hand, equation (42) has correctly incorporated the other components, including those due to $-\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}$, and $-\left(\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{2}} \psi_{\mathbf{0}}+\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{2}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}\right)$, in one fell swoop.

## Expansion for the $n=4$ Term:

For the $n=4$ case, we have

$$
\begin{equation*}
K_{4}=-\frac{1}{64}\left[\frac{1}{24}+\frac{1}{8}+\frac{1}{8}+\frac{1}{24}\right]=-\frac{1}{192}, \tag{53}
\end{equation*}
$$

and

$$
\begin{align*}
\frac{d^{4} H^{4}}{d z^{4}} & =\frac{d^{3}}{d z^{3}}\left[4 H^{3} \alpha_{1}\right] \\
& =4 \frac{d^{2}}{d z^{2}}\left[3 H^{2} \alpha_{1}^{2}+H^{3}\left[\frac{d \alpha_{1}}{d z}\right]\right] \\
& =4 \frac{d}{d z}\left[6 H \alpha_{1}^{3}+9\left[\frac{d \alpha_{1}}{d z}\right] \alpha_{1} H^{2}+\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right] H^{3}\right]  \tag{54}\\
& =24 \alpha_{1}^{4}+144\left[\frac{d \alpha_{1}}{d z}\right] \alpha_{1}^{2} H+36\left[\frac{d \alpha_{1}}{d z}\right]^{2} H^{2}+48\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right] \alpha_{1} H^{2}+4\left[\frac{d^{3} \alpha_{1}}{d z^{3}}\right] H^{3}
\end{align*}
$$

So in total,

$$
\begin{equation*}
I_{4}(z)=-\frac{1}{8} \alpha_{1}^{4}-\frac{3}{4}\left[\frac{d \alpha_{1}}{d z}\right] \alpha_{1}^{2} H-\frac{3}{16}\left[\frac{d \alpha_{1}}{d z}\right]^{2} H^{2}-\frac{1}{4}\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right] \alpha_{1} H^{2}-\frac{1}{48}\left[\frac{d^{3} \alpha_{1}}{d z^{3}}\right] H^{3} \tag{55}
\end{equation*}
$$

avoiding the replacement of $H$ with its explicit form this time around.

## Expansion for $n=5$ Term:

The $n=5$ expansion is

$$
\begin{align*}
I_{5}(z)= & \frac{1}{16} \alpha_{1}^{5}+\frac{5}{8}\left[\frac{d \alpha_{1}}{d z}\right] \alpha_{1}^{3} H+\frac{15}{32}\left[\frac{d \alpha_{1}}{d z}\right]^{2} \alpha_{1} H^{2}+\frac{5}{16}\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right] \alpha_{1}^{2} H^{2} \\
& +\frac{1}{24}\left[\frac{d^{3} \alpha_{1}}{d z^{3}}\right] \alpha_{1} H^{3}+\frac{5}{48}\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right]\left[\frac{d \alpha_{1}}{d z}\right] H^{3}+\frac{1}{384}\left[\frac{d^{4} \alpha_{1}}{d z^{4}}\right] H^{4}, \tag{56}
\end{align*}
$$

again omitting the replacement of $H$ with its explicit form for convenience.

## 4 Inherent Simplicities and Approximations

Comparing the expansion of equation (42) with the derived terms of the inverse series, it is clear that some of the terms are missing, and others have the wrong coefficients. It is important to be very clear regarding what has been "kept" of the full inverse series, and what has been "rejected", in utilizing equation (42). This section is concerned with developing both a clear sense of the approximations made in this simultaneous imaging and inversion formulation, and explicitly demonstrating the simplified role of certain classes of terms in the series for the 3rd and 4th order (a role that is assumed to continue at all orders).

### 4.1 Deriving $K_{n} d^{n} H^{n} / d z^{n}$ : Drop a Term, Find a Pattern

Equation (42) was deduced by noticing patterns in some components of the terms involving $\mathbf{V}_{\mathbf{1}}$ only; for instance, by paying attention to the $\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}$ term in the 3rd order equation, and ignoring terms like $\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{2}} \psi_{\mathbf{0}}$. In fact, it was designed initially to compute these terms only. We will demonstrate by considering $\alpha_{3}(z)$.

Set

$$
\begin{equation*}
\alpha_{3}(z)=I_{p}(z)+I_{s}(z), \tag{57}
\end{equation*}
$$

where

$$
\begin{align*}
I_{p}(-2 k) & =\frac{1}{4} k^{2} \int_{-\infty}^{\infty} e^{i 2 k z^{\prime}} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{z^{\prime}} \alpha_{1}\left(z^{\prime \prime}\right) \int_{-\infty}^{z^{\prime \prime}} \alpha_{1}\left(z^{\prime \prime \prime}\right) d z^{\prime \prime \prime} d z^{\prime \prime} d z^{\prime} \\
& +\frac{1}{4} k^{2} \int_{-\infty}^{\infty} \alpha_{1}\left(z^{\prime}\right) \int_{z^{\prime}}^{\infty} e^{i 2 k z^{\prime \prime}} \alpha_{1}\left(z^{\prime \prime}\right) \int_{-\infty}^{z^{\prime \prime}} \alpha_{1}\left(z^{\prime \prime \prime}\right) d z^{\prime \prime \prime} d z^{\prime \prime} d z^{\prime}  \tag{58}\\
& +\frac{1}{4} k^{2} \int_{-\infty}^{\infty} e^{i 2 k z^{\prime}} \alpha_{1}\left(z^{\prime}\right) \int_{-\infty}^{z^{\prime}} e^{-i 2 k z^{\prime \prime}} \alpha_{1}\left(z^{\prime \prime}\right) \int_{z^{\prime \prime}}^{\infty} e^{i 2 k z^{\prime \prime \prime}} \alpha_{1}\left(z^{\prime \prime \prime}\right) d z^{\prime \prime \prime} d z^{\prime \prime} d z^{\prime} \\
& +\frac{1}{4} k^{2} \int_{-\infty}^{\infty} \alpha_{1}\left(z^{\prime}\right) \int_{z^{\prime}}^{\infty} \alpha_{1}\left(z^{\prime \prime}\right) \int_{z^{\prime \prime}}^{\infty} e^{i 2 k z^{\prime \prime \prime}} \alpha_{1}\left(z^{\prime \prime \prime}\right) d z^{\prime \prime \prime} d z^{\prime \prime} d z^{\prime},
\end{align*}
$$

i.e. $I_{p}$ is due to $\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}$; $I_{s}$ is due to $\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{2}} \psi_{\mathbf{0}}$ and $\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{2}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}$, and is considered later. These terms are broken up based on the geometry of the scattering interactions. The third term in equation (58) involves a "down-up-down" scattering event, or a change in the directions of propagation; such expressions differ from those involving only upward scattering events, and don't immediately simplify in the same way. For the present analysis, we set this term to zero. Implementing the operator notation, the expression

$$
\begin{align*}
I_{p}(-2 k) & =\frac{1}{4} k^{2} \int_{-\infty}^{\infty} e^{i 2 k z^{\prime}} \alpha_{1}\left(z^{\prime}\right) \mathcal{H}\left\{\alpha_{1} H\left\{\alpha_{1}\right\}\right\} d z^{\prime} \\
& +\frac{1}{4} k^{2} \int_{-\infty}^{\infty} \alpha_{1}\left(z^{\prime}\right) \mathcal{H}^{-}\left\{e^{i 2 k z^{\prime}} \alpha_{1} \mathcal{H}\left\{\alpha_{1}\right\}\right\} d z^{\prime}  \tag{59}\\
& +\frac{1}{4} k^{2} \int_{-\infty}^{\infty} \alpha_{1}\left(z^{\prime}\right) \mathcal{H}^{-}\left\{\alpha_{1} \mathcal{H}^{-}\left\{e^{i 2 k z^{\prime}} \alpha_{1}\right\}\right\} d z^{\prime}
\end{align*}
$$

is produced. We proceed by making repeated use of the relationship in equation (20) in the definitions section, to produce from equation (59)

$$
\begin{align*}
I_{p}(-2 k) & =-\frac{1}{16}(i 2 k)^{2} \int_{-\infty}^{\infty} e^{i 2 k z^{\prime}} \alpha_{1}\left(z^{\prime}\right) \mathcal{H}\left\{\alpha_{1} \mathcal{H}\left\{\alpha_{1}\right\}\right\} d z^{\prime} \\
& -\frac{1}{16}(i 2 k)^{2} \int_{-\infty}^{\infty} e^{i 2 k z^{\prime}} \alpha_{1}\left(z^{\prime}\right) \mathcal{H}^{2}\left\{\alpha_{1}\right\} d z^{\prime}  \tag{60}\\
& -\frac{1}{16}(i 2 k)^{2} \int_{-\infty}^{\infty} e^{i 2 k z^{\prime}} \alpha_{1}\left(z^{\prime}\right) \mathcal{H}\left\{\alpha_{1} \mathcal{H}\left\{\alpha_{1}\right\}\right\} d z^{\prime}
\end{align*}
$$

noting that $-(1 / 16)(i 2 k)^{2}=(1 / 4) k^{2}$. Identifying these terms as Fourier transforms of second derivatives, equation (60) may be replaced with

$$
\begin{align*}
I_{p}(z) & =-\frac{1}{16} \frac{d^{2}}{d z^{2}}\left(\alpha_{1}(z) \mathcal{H}\left\{\alpha_{1} \mathcal{H}\left\{\alpha_{1}\right\}\right\}\right) \\
& -\frac{1}{16} \frac{d^{2}}{d z^{2}}\left(\alpha_{1}(z) \mathcal{H}^{2}\left\{\alpha_{1}\right\}\right)  \tag{61}\\
& -\frac{1}{16} \frac{d^{2}}{d z^{2}}\left(\alpha_{1}(z) \mathcal{H}\left\{\alpha_{1} \mathcal{H}\left\{\alpha_{1}\right\}\right\}\right),
\end{align*}
$$

and finally recalling that $\mathcal{H}_{2}\left\{\alpha_{1}\right\}=(1 / 2) H^{2}$, this becomes

$$
\begin{equation*}
I_{p}(z)=-\frac{1}{8} \frac{d^{2}}{d z^{2}}\left(\alpha_{1}(z) \mathcal{H}^{2}\left\{\alpha_{1}\right\}\right)=-\frac{1}{24} \frac{d^{3}}{d z^{3}}\left(H^{3}\right) \tag{62}
\end{equation*}
$$

The preceding gives one a sense of how equation (42) was developed - doing this for a number of orders, and watching how the coefficient was generated. Interestingly, equation (62) is the negative of the associated $I_{3}(z)$ :

$$
\begin{equation*}
I_{p}(z)=-K_{3} \frac{d^{3} H^{3}}{d z^{3}}=-I_{3}(z) \tag{63}
\end{equation*}
$$

and hence, using equation (51), is

$$
\begin{equation*}
I_{p}(z)=-\frac{1}{4} \alpha_{1}^{3}-\frac{3}{4}\left[\frac{d \alpha_{1}}{d z}\right] \alpha_{1} H-\frac{1}{8}\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right] H^{2} . \tag{64}
\end{equation*}
$$

The difference between $I_{3}$ and the third order series terms of equation (38) indicates the effect of dropping the term from equation (58).

### 4.2 Simplifying Groups of Terms

Any reader who has taken the trouble to write down the inverse scattering series terms (even in general operator form), beyond the third or fourth order, say, would be excused if they looked at equation (42) with mounting skepticism. Its implied simplicity seems to contradict the growing complexity of the interactions of the series with increasing order. In this part of the paper, we attempt to reconcile this apparent contradiction by explicitly considering terms in third and fourth order. We use the results of this investigation to deduce a pattern of behaviour for the series, a pattern that we assume holds at all orders.

Shaw et al. (2003) point out that the LOIS terms are contributed-to from both terms that are in $\mathbf{V}_{\mathbf{1}}$ only (e.g. $\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}$ ), and terms that are in $\mathbf{V}_{\mathbf{1}}, \mathbf{V}_{\mathbf{2}}, \ldots$ etc. (e.g. $\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{2}} \psi_{\mathbf{0}}$ ). Here we call these "primary" terms and "secondary" terms respectively ${ }^{2}$.

[^3]The fact that the LOIS is reproduced in the expansion of equation (42) for the third order (this can be seen by comparing equations (41) and (52)) suggests that simultaneous imaging and inversion correctly incorporates both primary and secondary terms. But we have just shown that equation (42) is almost exactly the same as the terms due to $\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}$ only - primary terms. The only way all these statements make sense is if the secondary terms are bringing about simple, predictable, alterations to their associated primary terms.

## Secondary terms for 3rd Order

Consider again the 3rd order term in which

$$
\begin{equation*}
\alpha_{3}(z)=I_{p}(z)+I_{s}(z), \tag{65}
\end{equation*}
$$

for primary and secondary terms respectively (that is, $I_{p}(z)$ is due to $-\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}$ and $I_{s}(z)$ is due to $\left.-\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{2}} \psi_{\mathbf{0}}-\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{2}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}\right)$. In the third order, the difference between the primary term due to $\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}$ and the full expression of equation (42) is a minus sign - see equation (63). However, since the secondary terms only have the power to change the primary term by adding something to $i t$, the simple process of changing a sign can only be achieved by constructing twice the negative of $I_{3}(z)$ - a fair amount of work. We can see this by computing $I_{s}(z)$ from equation (57). Writing down equation (47), generalized to accommodate $\alpha_{1}$ and $\alpha_{2}$, and substituting the expression for $\alpha_{2}$ therein, one has

$$
\begin{align*}
I_{s}(z) & =-\frac{1}{2}\left[\frac{d \alpha_{1}}{d z}\right] \mathcal{H}\left\{\alpha_{2}\right\}-\alpha_{1} \alpha_{2}-\frac{1}{2}\left[\frac{d \alpha_{2}}{d z}\right] \mathcal{H}\left\{\alpha_{1}\right\} \\
& =-\frac{1}{2}\left[\frac{d \alpha_{1}}{d z}\right] \mathcal{H}\left\{K_{2} \frac{d^{2} H^{2}}{d z^{2}}\right\}-\alpha_{1}\left[K_{2} \frac{d^{2} H^{2}}{d z^{2}}\right]-\frac{1}{2}\left[K_{2} \frac{d^{3} H^{2}}{d z^{3}}\right] \mathcal{H}\left\{\alpha_{1}\right\} \\
& =-K_{2}\left(\frac{1}{2}\left[\frac{d \alpha_{1}}{d z}\right]\left[\frac{d H^{2}}{d z}\right]+\alpha_{1} \frac{d}{d z}\left[2 H \alpha_{1}\right]+\frac{1}{2} \frac{d^{2}}{d z^{2}}\left[2 H \alpha_{1}\right] H\right) \\
& =-K_{2}\left(\left[\frac{d \alpha_{1}}{d z}\right] \alpha_{1} H+2 \alpha_{1}\left[\alpha_{1}^{2}+\left[\frac{d \alpha_{1}}{d z}\right] H\right]+\frac{d}{d z}\left[\alpha_{1}^{2}+\left[\frac{d \alpha_{1}}{d z}\right] H\right] H\right)  \tag{66}\\
& =-K_{2}\left(3\left[\frac{d \alpha_{1}}{d z}\right] \alpha_{1} H+2 \alpha_{1}^{3}+2\left[\frac{d \alpha_{1}}{d z}\right] \alpha_{1} H+\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right] H^{2}+\left[\frac{d \alpha_{1}}{d z}\right] \alpha_{1} H\right) \\
& =-K_{2}\left(6\left[\frac{d \alpha_{1}}{d z}\right] \alpha_{1} H+2 \alpha_{1}^{3}+\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right] H^{2}\right) \\
& =\frac{1}{2} \alpha_{1}^{3}+\frac{3}{2}\left[\frac{d \alpha_{1}}{d z}\right] \alpha_{1} H+\frac{1}{4}\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right] H^{2},
\end{align*}
$$

which is twice $-I_{p}(z)$. (Note that the constant $K_{2}$ may be brought out of the $\mathcal{H}\{\cdot\}$ operator.) This result supports the postulate that the secondary terms effect some simple alteration to
the primary terms. It seems reasonable to look for this behaviour in the other, higher order, terms as well.

## Secondary terms for 4th Order

We proceed by considering the secondary components of the fourth order term. Since $\mathbf{V}_{\mathbf{4}}$ is solved-for through the equation

$$
\begin{align*}
\mathbf{G}_{0} \mathbf{V}_{\mathbf{4}} \psi_{\mathbf{0}}= & -\mathbf{G}_{0} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}} \\
& -\left(\mathbf{G}_{0} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{2}} \psi_{\mathbf{0}}+\mathbf{G}_{0} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{2}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}+\mathbf{G}_{0} \mathbf{V}_{\mathbf{2}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}\right)  \tag{67}\\
& -\left(\mathbf{G}_{0} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{3}} \psi_{\mathbf{0}}+\mathbf{G}_{0} \mathbf{V}_{\mathbf{3}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}+\mathbf{G}_{0} \mathbf{V}_{\mathbf{2}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{2}} \psi_{\mathbf{0}}\right),
\end{align*}
$$

there are now six secondary components that need attention. Computation of the secondary terms in this fourth order set is different from that of the third order, because in this case incomplete terms are substituted in, namely the approximation $I_{3} \approx \alpha_{3}$. Previously, i.e. in equation (66), only substitution for $\alpha_{2}$, which is fully expressed by equation (42), was needed. Not so this time.

Calling the inversion terms counterpart to those of equation (67) respectively

$$
\begin{align*}
\alpha_{4}= & \text { PRIMARY } \\
& +I_{112}+I_{121}+I_{211}  \tag{68}\\
& +I_{13}+I_{31}+I_{22},
\end{align*}
$$

We use the same approach as that taken in equations (58) - (61) to find expressions for the six secondary terms above. At the risk of repeating ourselves, terms like the third of four in equation (58) are neglected in equation (68); this means that not only are they not included in the primary terms, but also they are not ever substituted into the secondary terms where they'll be expected, such as in a $\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{3}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}$-type expression.

In appendix A , we show that, after neglecting such terms,

$$
\begin{equation*}
I_{112}+I_{121}+I_{211}+I_{13}+I_{31}+I_{22}=0 . \tag{69}
\end{equation*}
$$

That is, through a careful balancing, the fourth order secondary terms, none of which are zero on their own, sum to nil.

To summarize, in spite of considerable effort, these "secondary" components of the inverse series with one quarter of the scattering interactions neglected (the neglecting occurs in going from equation (58) to (59)) only either change the sign of their associated primary terms, as in the third order, or do nothing, as in the fourth. This is why equation (42), although
designed to compute only primary terms, produces the output of both primary and secondary terms with the simple inclusion of a factor which alters the sign of the output. At present we assume that the patterns seen explicitly, thus far, hold for all the orders of this portion of the inverse series. The correct reproduction of the LOIS subseries found by Shaw et al. (2003), from equation (42), would be a good check of this assumption.

## 5 Associated Subseries

In this section we look more carefully at some of the subseries which arise when equation (42) is expanded for $n$ orders. Some of these subseries have the form of the pure imaging and pure inversion type tasks, which have been developed and discussed elsewhere.

### 5.1 Leading Order Imaging Subseries

Previously, a set of $I_{n}(z)$ 's were explicitly computed and/or listed. Notice that if we collect and sum the last term in each:

$$
\begin{equation*}
\alpha_{1}-\frac{1}{2}\left[\frac{d \alpha_{1}}{d z}\right] H+\frac{1}{8}\left[\frac{d^{2} \alpha_{1}}{d z^{2}}\right] H^{2}-\frac{1}{48}\left[\frac{d^{3} \alpha_{1}}{d z^{3}}\right] H^{3}+\ldots \tag{70}
\end{equation*}
$$

the LOIS series of Shaw et al. (2004) is reproduced:

$$
\begin{equation*}
\alpha_{L O I S}=\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1} \frac{1}{(k-1)!}\left[\frac{d^{k} \alpha_{1}}{d z^{k}}\right] H^{k} . \tag{71}
\end{equation*}
$$

Equation (42) is therefore computing terms at all orders which are combinations of primary and secondary components. This supports the idea that $(i)$ equation (42) is capturing much of the behaviour of the series, and (ii) therefore the net effect of secondary terms on this portion of the series is the aforementioned change (or not change) of sign.

### 5.2 Inversion Subseries

Next, notice that if the first terms in each of the expansions above are summed, one gets

$$
\begin{equation*}
\alpha_{1}-\frac{1}{2} \alpha_{1}^{2}+\frac{1}{4} \alpha_{1}^{3}-\frac{1}{8} \alpha_{1}^{4}+\frac{1}{16} \alpha_{1}^{5}+\ldots \tag{72}
\end{equation*}
$$

a series which, because of the form of its constituents, must be devoted to inversion tasks. This is partial inversion only, since as discussed, some terms were neglected in the derivation of equation (42). Nevertheless there is a pattern:

$$
\begin{equation*}
\alpha_{P I N V}(z)=\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1} \alpha_{1}^{k}(z) . \tag{73}
\end{equation*}
$$

The question is, what does this series do? The full inversion subseries is

$$
\begin{equation*}
\alpha_{I N V}=\alpha_{1}-\frac{1}{2} \alpha_{1}^{2}+\frac{3}{16} \alpha_{1}^{3}-\ldots \tag{74}
\end{equation*}
$$

such that, for 1D reflection over a single interface with a reflection coefficient of $R_{1}$, the inverted perturbation amplitude becomes, in terms of this $R_{1}$,

$$
\begin{equation*}
\alpha_{I N V}=4 R_{1}\left(1-2 R_{1}+3 R_{1}^{2}-4 R_{1}^{3}+\ldots\right)=\frac{4 R_{1}}{\left(1+R_{1}\right)^{2}} \tag{75}
\end{equation*}
$$

Clearly equation (73) is not this same series, therefore it will not produce the correct results. To find out what results are produced, we set a similar problem up, i.e. a 1D reflection experiment over a single interface. No imaging will be required, so omitting all $z$ dependence, a Born approximate amplitude for the contrast of

$$
\begin{equation*}
\alpha_{1}=4 R_{1} \tag{76}
\end{equation*}
$$

is produced. Substituting this into the partial inversion subseries, for $\alpha_{P I N V}$ in equation (73) produces

$$
\begin{equation*}
\alpha_{P I N V}=4 R_{1}\left(1-2 R_{1}+4 R_{1}^{2}-8 R_{1}^{3}+16 R_{1}^{4}-\ldots\right), \tag{77}
\end{equation*}
$$

so up to second order in $R_{1}$ the partial inversion subseries is the same as the full; compare equation (75) with (77). Beyond that they begin to differ, more-so with higher order in $R_{1}$. This begins to suggest that the partial inversion subseries ( $\alpha_{P I N V}$ ) is a low $R_{1}$ approximation. Of course, without infinite terms, so is the full! This means that at low $R_{1}$ both the full and partial inversion series are equivalent, but at higher $R_{1}$ they differ, with the full inversion series performing better than the partial. This is demonstrated in a series of plots of $\alpha_{I N V}$ and $\alpha_{P I N V}$ against $R_{1}$, in Figure 1. The partial subseries seems to capture the true reflection coefficient very well up until approximately $R_{1}=0.4$ over 8 terms. It seems that $\alpha_{\text {PINV }}$ converges to something very close to $\alpha_{I N V}$, but more slowly as $R_{1}$ increases.

The point is that the portion of the full inverse scattering series that come from the neglected terms appear to supply - in part - the inversion subseries with higher-order components. At low order, the partial inversion subseries within equation (42) and the full inversion subseries are almost equivalent.


Figure 1: Perturbations $\alpha_{I N V} / \alpha_{P I N V}$ vs. $R_{1}$. Comparison of full (dashed) and partial (solid) inversion subseries for single interface experiment using progressively more terms (dotted line is the true model). Both are low $R_{1}$ approximations, but we know that in its totality the full inversion converges to the true model. (a) 1 term, (b) 2 terms, (c) 3 terms, (d) 8 terms. The partial inversion subseries follows the full subseries well for $R_{1}$ values under 0.4 after 8 terms.

## 6 Alternative Mathematical Forms

The simultaneous imaging and inversion form of equation (42) may equivalently be expressed

$$
\begin{equation*}
\alpha_{S I I}(z)=\sum_{n=0}^{\infty} \frac{(-1 / 2)^{n}}{n!} \frac{d^{n}}{d z^{n}}\left[\alpha_{1}(z) H^{n}\right] . \tag{78}
\end{equation*}
$$

Using the forward and inverse Fourier transform,

$$
\begin{align*}
\alpha_{S I I}(z) & =\frac{1}{2 \pi} \sum_{n=0}^{\infty} \frac{(-1 / 2)^{n}}{n!} \int_{-\infty}^{\infty} e^{i k z}\left[(-i k)^{n} \int_{-\infty}^{\infty} e^{-i k z^{\prime}}\left[\alpha_{1}\left(z^{\prime}\right) H^{n}\right]\left(z^{\prime}\right) d z^{\prime}\right] d k \\
& =\frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i k\left(z-z^{\prime}\right)} \alpha_{1}\left(z^{\prime}\right)\left[\sum_{n=0}^{\infty} \frac{\left[-\frac{i k}{2} H\left(z^{\prime}\right)\right]^{n}}{n!}\right] d z^{\prime} d k  \tag{79}\\
& =\frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{\infty}^{\infty} e^{i k\left(z-z^{\prime}\right)} \alpha_{1}\left(z^{\prime}\right) e^{\frac{i k}{2} H\left(z^{\prime}\right)} d z^{\prime} d k \\
& =\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i k z}\left[\int_{\infty}^{\infty} e^{-i k\left[z^{\prime}-\frac{1}{2} H\left(z^{\prime}\right)\right]} \alpha_{1}\left(z^{\prime}\right) d z^{\prime}\right] d k
\end{align*}
$$

having made use of

$$
\begin{equation*}
e^{x}=\sum_{n=0}^{\infty} \frac{x^{n}}{n!} . \tag{80}
\end{equation*}
$$

In other words we may interpret $\alpha_{\text {SII }}(z)$ as being the inverse Fourier transform of $\alpha_{1}$ after $\alpha_{1}(z)$ has undergone a Fourier-like transformation where the kernel,

$$
\begin{equation*}
\exp \left\{-i k\left[z^{\prime}-\frac{1}{2} \int_{0}^{z^{\prime}} \alpha_{1}\left(z^{\prime \prime}\right) d z^{\prime \prime}\right]\right\} \tag{81}
\end{equation*}
$$

depends on the integral of $\alpha_{1}(z)$.
As an aside, in Figure 2 we demonstrate the numerical computation of this closed form solution for three numerical models which will appear again in Innanen et al. (2004) in this report (see that paper for model details and further numerical details of such reconstructions). In solid black is the reconstruction, overlying the true perturbation $\alpha(z)$; contrasting these, the Born approximation $\alpha_{1}(z)$ is displayed as a dotted line. The reconstruction has captured the expected location and amplitude of the true contrasts for all three examples (not to mention a smoothness that is probably the result of the crude quadrature routine used to compute the Fourier-like integral; the authors continue to investigate).

## 7 Coupled Imaging-Inversion: Intuitive Interpretation

One of the keys to pursuing (eventual) practical implementation of algorithms based on inverse scattering is to have a clear understanding of the operations being visited upon the data as a result of the theory. Such a working knowledge advances our general understanding of how amplitude and timing information is extracted and synthesized in the theoretical milieu of the inverse scattering series. In this section we explore the numerical action of


Figure 2: Closed form for the simultaneous imaging and inversion algorithm, illustrated numerically on three input models (see Innanen et al. (2004) for more detail on these input models). In solid black is the reconstruction, overlying the true perturbation $\alpha(z)$; contrasting these, the Born approximation $\alpha_{1}(z)$ is displayed as a dotted line.
equation (42) on a general input, with the aim of gaining a clearer signal processing-based view of what simultaneous imaging and inversion does.
The engine of simultaneous imaging and inversion is

$$
\begin{equation*}
\frac{d^{n}(\cdot)^{n}}{d z^{n}} \tag{82}
\end{equation*}
$$

where the quantity in brackets $(\cdot)$ is a discontinuous input akin to the second integral of a seismic trace ( $H$ in this paper). In words, the quantity $(\cdot)$ is taken to the $n$ 'th power, and the $n$ 'th derivative with respect to $z$ is carried out upon the result. This is done for a (large) range of values of $n$, and the various outputs are weighted and summed.
The two cascaded operations, (1) take the n'th power and (2) take the n'th derivative, have a variable impact upon the integral of the input Born approximation. This helps explain
how the processes of imaging and inversion can proceed simultaneously through a simple computation.

We consider synthetic data due to a 1D normal incidence model. In pseudo-depth the resulting trace is a series of discrete impulses. This is integrated once to produce $\alpha_{1}(z)$, and again to produce $H$. In general for piecewise constant impedance Earth models, the form of $H$ tends, therefore, to be a piecewise linear signal. The key is to follow what the cascaded operation $d^{n} H^{n} / d z^{n}$ does to such an input $H$.
$H$ has two distinct types of structure, each of which reacts very differently to this operator. First consider an element of $H$ away from all discontinuities. In general elements of $H$, called, say, $H_{l i n}(z)$, on such an interval may be described, as a general linear function, by

$$
\begin{equation*}
H_{l i n}(z)=a z+b \tag{83}
\end{equation*}
$$

where $a$ and $b$ are some constants determined by the data at and above $z_{1}$. (Notice that if $z_{1}$ is the location of the shallowest interface, with a reflection coefficient of $R_{1}$, then $a=4 R_{1}$.) Computing the first four orders of the operator $K_{n} d^{n} H_{l i n}^{n} / d z^{n}$ gives

$$
\begin{align*}
K_{1} \frac{d H_{l i n}(z)}{d z} & =a  \tag{84}\\
( & \left.=4 R_{1}\right)
\end{align*}
$$

$$
\begin{align*}
K_{2} \frac{d^{2} H_{l i n}^{2}(z)}{d z^{2}} & =-\frac{1}{4} \frac{d^{2}}{d z^{2}}\left[a^{2} z^{2}+2 a z b+b^{2}\right] \\
& =-\frac{1}{2} a^{2}  \tag{85}\\
& \left(=-8 R_{1}^{2}\right)
\end{align*}
$$

$K_{3} \frac{d^{3} H_{l i n}^{3}(z)}{d z^{3}}=\frac{1}{24} \frac{d^{3}}{d z^{3}}\left[a^{3} z^{3}+\ldots\right]$
$=\frac{1}{4} a^{3}$

$$
\begin{equation*}
\left(=16 R_{1}^{3}\right), \tag{86}
\end{equation*}
$$

$$
\begin{align*}
K_{4} \frac{d^{4} H_{l i n}^{4}(z)}{d z^{4}} & =-\frac{1}{192} \frac{d^{4}}{d z^{4}}\left[a^{4} z^{4}+\ldots\right] \\
& =-\frac{1}{8} a^{4}  \tag{87}\\
& \left(=-32 R_{1}^{4}\right)
\end{align*}
$$

where in brackets the particular value $a=4 R_{1}$ is used. It is clear that, at all orders, the operator $d^{n}(\cdot)^{n} / d z^{n}$ does not take these linear features $H_{\text {lin }}$ and create an output with any complicated spatial structure. In fact, the $n$ 'th power and the $n$ 'th derivative counteract each other almost completely for an input with a linear dependence. $K_{n} d^{n}\left(H_{l i n}\right)^{n} / d z^{n}$ is the transformation from a linear function to a constant function, in which the only important task of $n$ is in determining the weight - or the eventual value - of the constant output.

So equation (42) in fact operates quite gently on the second integral of the data away from its discontinuities. It acts to alter its amplitudes in a way that corresponds exactly to the partial inversion subseries discussed earlier in this paper. To see this, compare the bracketed results of equations (84) - (87), for the single interface case, with the terms of this partial inversion subseries for the same case in equation (77).

But the genteel act of transforming a linear function into a constant function belies the volatility of the operator that performs it. When the operator encounters the second structural type found in $H$-like inputs, at and near its discontinuities, things change drastically.
$d^{n}(\cdot)^{n} / d z^{n}$ actually behaves like an edge-detector - it tends to do little to portions of a function that are well approximated by low-order polynomials, while strongly reacting to portions that resemble high-order polynomials, and especially signal edges and discontinuities. In the case of $H$-like functions, these discontinuities are at points where piecewise linear functions conjoin. In Figure 3 a test discontinuity (a) of this kind is operated on by $d^{n}(\cdot)^{n} / d z^{n}$, again for $n=1-4$. Each of Figures $3 \mathrm{a}-\mathrm{d}$ has 2 panels; the top is the function of Figure 3a taken to the $n$ 'th power, and the bottom is the numerical $n$ 'th derivative thereof.

Interestingly, what is returned is a signal with the characteristics of weighted derivatives of the original function. This is not necessarily an intuitive result, since by eye, the discontinuities associated with Figures 3a-d, top panels, appear to be of changing order (or regularity). However, numerically, the output continues reflect a "triangle"-like discontinuity, i.e. of order 1. What changes is that the differences in the slope on either side of the discontinuity become much larger, and this produces the effective weights on the output derivatives.

So, near the discontinuities that typify the integral of the Born approximation, $d^{n}(\cdot)^{n} / d z^{n}$ is outputting the sum of weighted derivatives of a piecewise constant function, 0 'th (Figure 3a, bottom panel) through 3rd (Figure 3d, bottom panel) in these examples. This is in agreement with the mechanics of the leading order imaging subseries as investigated in Shaw et al. (2003), and seen in equation (71) in this paper.

In summary, the simultaneous imaging and inversion formula can be seen to act as a flexible operator that mimics both the inversion subseries and the leading order imaging subseries. One dominates over the other based on the proximity of the operator to discontinuities in the integral of the Born approximation.


Figure 3: Illustration of the effect of the operator $d^{n}(\cdot)^{n} / d z^{n}$ on the $n^{\prime}$ th power of the characteristic discontinuities of the input $H$ (or second integral of the data). Each of the four examples (a)-(d) has two panels. Top panel: $n$ 'th power of input $H$; bottom panel: $n$ 'th numerical derivative of the top panel. This is done for four example orders: (a) 1, (b) 2, (c), 3, and (d) 4. In spite of the increasing curvature on either side of the discontinuity, numerically the results consistently resemble weighted $n$ 'th derivatives of order 1 discontinuities.

## 8 Conclusions

We present an analysis of an algorithm that results from a purposeful coupling of terms in the inverse scattering series that concern the processing and inversion of seismic primaries. The interest lies principally in the fact that the coupling represents such a large fraction of the terms in the series that "remain" after multiples have been eliminated: the consequent algorithm and its properties speak volumes about the overall behaviour of the series.

The simplicity of form that results from a particular choice of inverse scattering series terms is developed, and the extent of the approximation of the full series is investigated - it is noted that what is missing effectively involves higher order imaging and inversion terms.

As such this "simultaneous imaging and inversion" procedure is very accurate for low and intermediate contrast levels, with some error accumulating at very high contrast.

We then consider a second analytic form for the 1D normal incidence acoustic algorithm, which permits a collapse to closed form (again for this simplified instance).

Finally, we attempt to describe the behaviour of this algorithm from a signal processing point of view, in particular with an aim to understand how a simple operator can perform such varied tasks (reflector location and target identification) at the same time. The answer is that the operator responds flexibly to the structure it encounters, behaving strongly at discontinuities and weakly away from them, to respectively move them in depth or merely change their amplitude.

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## Appendix A: Inverse Scattering Terms

We show that the chosen terms from the fourth order imaging/inversion subseries (for 1D constant density acoustic imaging and wavespeed inversion) sum to nil. These components are

$$
\begin{equation*}
I_{112}+I_{121}+I_{211}+I_{13}+I_{31}+I_{22} \tag{88}
\end{equation*}
$$

As in the derivations of this paper, the operators $\mathcal{H}$ and the derivatives, that appear here arise after substitution of the form of the Green's functions into the terms of the inverse scattering series. They are due, respectively, to the alterations of the integration limits and the appearance of powers of the wavenumber. To start,

$$
\begin{align*}
& I_{112}=-\frac{1}{16} \frac{d^{2}}{d z^{2}}\left[\alpha_{2} \mathcal{H}\left\{\alpha_{1} \mathcal{H}\left\{\alpha_{1}\right\}\right\}+\alpha_{1} \mathcal{H}\left\{\alpha_{1}\right\} \mathcal{H}\left\{\alpha_{2}\right\}+\alpha_{1} \mathcal{H}\left\{\alpha_{1} \mathcal{H}\left\{\alpha_{2}\right\}\right\}\right] \\
& I_{121}=-\frac{1}{16} \frac{d^{2}}{d z^{2}}\left[\alpha_{1} \mathcal{H}\left\{\alpha_{2} \mathcal{H}\left\{\alpha_{1}\right\}\right\}+\alpha_{2} \mathcal{H}\left\{\alpha_{1}\right\} \mathcal{H}\left\{\alpha_{1}\right\}+\alpha_{1} \mathcal{H}\left\{\alpha_{2} \mathcal{H}\left\{\alpha_{1}\right\}\right\}\right]  \tag{89}\\
& I_{211}=-\frac{1}{16} \frac{d^{2}}{d z^{2}}\left[\alpha_{1} \mathcal{H}\left\{\alpha_{1} \mathcal{H}\left\{\alpha_{2}\right\}\right\}+\alpha_{1} \mathcal{H}\left\{\alpha_{1}\right\} \mathcal{H}\left\{\alpha_{2}\right\}+\alpha_{2} \mathcal{H}\left\{\alpha_{1} \mathcal{H}\left\{\alpha_{1}\right\}\right\}\right]
\end{align*}
$$

so

$$
\begin{align*}
I_{112}+I_{121}+I_{211}= & I_{3 V}=-\frac{1}{16} \frac{d^{2}}{d z^{2}}\left[2 \alpha_{2} \mathcal{H}\left\{\alpha_{1} \mathcal{H}\left\{\alpha_{1}\right\}\right\}+2 \alpha_{1} \mathcal{H}\left\{\alpha_{1}\right\} \mathcal{H}\left\{\alpha_{2}\right\}\right.  \tag{90}\\
& \left.+2 \alpha_{1} \mathcal{H}\left\{\alpha_{1} \mathcal{H}\left\{\alpha_{2}\right\}\right\}+2 \alpha_{1} \mathcal{H}\left\{\alpha_{2} \mathcal{H}\left\{\alpha_{1}\right\}\right\}+\alpha_{2} \mathcal{H}\left\{\alpha_{1}\right\} \mathcal{H}\left\{\alpha_{1}\right\}\right] .
\end{align*}
$$

Substituting appropriate versions of equation (42) into the above produces

$$
\begin{align*}
I_{3 V}= & -\frac{1}{16} \frac{d^{2}}{d z^{2}}\left[2\left(K_{2} \frac{d^{2} H^{2}}{d z^{2}}\right) \mathcal{H}\left\{\alpha_{1} \mathcal{H}\left\{\alpha_{1}\right\}\right\}+2 \frac{d H}{d z} \mathcal{H}\left\{\alpha_{1}\right\} \mathcal{H}\left\{K_{2} \frac{d^{2} H^{2}}{d z^{2}}\right\}\right. \\
& \left.+2 \frac{d H}{d z} \mathcal{H}\left\{\alpha_{1} \mathcal{H}\left(K_{2} \frac{d^{2} H^{2}}{d z^{2}}\right)\right\}+2 \frac{d H}{d z} \mathcal{H}\left\{K_{2} \frac{d^{2} H^{2}}{d z^{2}} H\right\}+K_{2} \frac{d^{2} H^{2}}{d z^{2}} H^{2}\right] \\
= & -\frac{K_{2}}{16} \frac{d^{2}}{d z^{2}}\left[\frac{d^{2} H^{2}}{d z^{2}} H^{2}+2 \frac{d H}{d z} \frac{d H^{2}}{d z} H+2 \frac{d H}{d z} \mathcal{H}\left\{\frac{d H}{d z} \frac{d H^{2}}{d z}\right\}\right. \\
& \left.+2 \frac{d H}{d z} \mathcal{H}\left\{\frac{d}{d z}\left(2 \frac{d H}{d z} H\right)\right\}+\frac{d}{d z}\left(2 \frac{d H}{d z} H\right) H^{2}\right] \\
= & -\frac{K_{2}}{16} \frac{d^{2}}{d z^{2}}\left[4 H^{3} \frac{d^{2} H}{d z^{2}}+8\left(\frac{d H}{d z} H\right)^{2}+8 \frac{d H}{d z} \mathcal{H}\left\{\left(\frac{d H}{d z}\right)^{2} H\right\}\right. \\
& \left.+4 \frac{d H}{d z} \mathcal{H}\left\{\frac{d^{2} H}{d z^{2}} H^{2}\right\}\right] \\
= & -\frac{K_{2}}{16} \frac{d^{2}}{d z^{2}}\left[4 H^{3} \frac{d^{2} H}{d z^{2}}+8\left(\frac{d H}{d z} H\right)^{2}+8 \frac{d H}{d z}\left(\frac{1}{2} \frac{d H}{d z} H^{2}\right)\right.  \tag{91}\\
& \left.-8 \frac{d H}{d z} \mathcal{H}\left\{\frac{1}{2} \frac{d^{2} H}{d z^{2}} H^{2}\right\}+4 \frac{d H}{d z} \mathcal{H}\left\{\frac{d^{2} H}{d z^{2}} H^{2}\right\}\right] \\
= & -\frac{K_{2}}{16} \frac{d^{2}}{d z^{2}}\left[4 \frac{d^{2} H}{d z^{2}} H^{3}+12\left(\frac{d H}{d z} H\right)^{2}\right] \\
= & -\frac{K_{2}}{4} \frac{d^{2}}{d z^{2}}\left[\frac{d}{d z}\left(\frac{d H}{d z} H^{3}\right)-3 H^{2}\left(\frac{d H}{d z}\right)^{2}+3 H^{2}\left(\frac{d H}{d z}\right)^{2}\right] \\
= & -\frac{K_{2}}{4} \frac{d^{3}}{d z^{3}}\left[\frac{d H}{d z} H^{3}\right] \\
= & -\frac{K_{2}}{16} \frac{d^{4} H^{4}}{d z^{4}} .
\end{align*}
$$

The terms $I_{31}+I_{13}$ are of the same form as the secondary terms of the third order case, i.e. equation (66):

$$
\begin{align*}
I_{31}+I_{13} & =-\frac{1}{2} \frac{d}{d z}\left[\frac{d H}{d z} H\left\{\alpha_{3}\right\}+\alpha_{3} H\right] \\
& =-\frac{K_{3}}{2} \frac{d^{2}}{d z^{2}}\left[\frac{d^{2} H^{3}}{d z^{2}} H\right] \\
& =-\frac{3 K_{3}}{2} \frac{d^{2}}{d z^{2}}\left[2\left(\frac{d H}{d z} H\right)^{2}+H^{3} \frac{d^{2} H}{d z^{2}}\right] \\
& =-\frac{3 K_{3}}{2} \frac{d^{2}}{d z^{2}}\left[2\left(\frac{d H}{d z} H\right)^{2}+\frac{d}{d z}\left(\frac{d H}{d z} H^{3}\right)-3\left(\frac{d H}{d z} H\right)^{2}\right]  \tag{92}\\
& =\frac{3 K_{3}}{2} \frac{d^{2}}{d z^{2}}\left[\left(\frac{d H}{d z} H\right)^{2}-\frac{d}{d z}\left(\frac{d H}{d z} H^{3}\right)\right] \\
& =-\frac{3 K_{3}}{8} \frac{d^{4} H^{4}}{d z^{4}}+\frac{3 K_{3}}{2} \frac{d^{2}}{d z^{2}}\left[\left(\frac{d H}{d z} H\right)^{2}\right] .
\end{align*}
$$

Similarly compute $I_{22}$ :

$$
\begin{align*}
I_{22} & =-\frac{1}{2} \frac{d}{d z}\left[\alpha_{2} \mathcal{H}\left\{\alpha_{2}\right\}\right] \\
& =-\frac{1}{2} \frac{d}{d z}\left[K_{2} \frac{d^{2} H^{2}}{d z^{2}} \mathcal{H}\left\{K_{2} \frac{d^{2} H^{2}}{d z^{2}}\right\}\right] \\
& =-\frac{K_{2}^{2}}{2} \frac{d}{d z}\left[\frac{d^{2} H^{2}}{d z^{2}} \frac{d H^{2}}{d z}\right]  \tag{93}\\
& =-K_{2}^{2} \frac{d^{2}}{d z^{2}}\left[\left(\frac{d H}{d z} H\right)^{2}\right] .
\end{align*}
$$

Summing equations (92) and (93), we have

$$
\begin{align*}
I_{31}+I_{13}+I_{22} & =-\frac{3 K_{3}}{8} \frac{d^{4} H^{4}}{d z^{4}}+\frac{3 K_{3}}{2} \frac{d^{2}}{d z^{2}}\left[\left(\frac{d H}{d z} H\right)^{2}\right]-K_{2}^{2} \frac{d^{2}}{d z^{2}}\left[\left(\frac{d H}{d z} H\right)^{2}\right] \\
& =-\frac{3 K_{3}}{8} \frac{d^{4} H^{4}}{d z^{4}}+\left(\frac{3}{2} K_{3}-K_{2}^{2}\right) \frac{d^{2}}{d z^{2}}\left[\left(\frac{d H}{d z} H\right)^{2}\right] . \tag{94}
\end{align*}
$$

However,

$$
\begin{equation*}
\frac{3}{2} K_{3}-K_{2}^{2}=\frac{3}{2}\left(\frac{1}{24}\right)-\left(-\frac{1}{4}\right)^{2}=\frac{1}{16}-\frac{1}{16}=0 \tag{95}
\end{equation*}
$$

So

$$
\begin{equation*}
I_{31}+I_{13}+I_{22}=-\frac{3 K_{3}}{8} \frac{d^{4} H^{4}}{d z^{4}} . \tag{96}
\end{equation*}
$$

Equations (91) and (96) now contain the totality of the fourth order secondary terms; these are summed, and the result is added to the primary term (due to $\mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \mathbf{G}_{\mathbf{0}} \mathbf{V}_{\mathbf{1}} \psi_{\mathbf{0}}$ ) to produce $\alpha_{4}$. But summing these secondary terms produces

$$
\begin{equation*}
I_{3 V}+I_{31}+I_{13}+I_{22}=-\frac{d^{4} H^{4}}{d z^{4}}\left(\frac{K_{2}}{16}+\frac{3 K_{3}}{8}\right)=-\frac{d^{4} H^{4}}{d z^{4}}\left(-\frac{1}{64}+\frac{1}{64}\right)=0 \tag{97}
\end{equation*}
$$

# Basic numerics in the nonlinear inversion of primaries: simultaneous imaging and inversion II 

Kristopher A. Innanen, Arthur B. Weglein and Tad J. Ulrych


#### Abstract

The coupling of the tasks of imaging and inverting seismic primaries discussed in Innanen et al. (2004) (see also Innanen, 2003 and Innanen and Weglein, 2003), while from a practical viewpoint running contrary to the strategy of task-separation (e.g. Weglein et al., 2003), is valuable as a tool for the understanding of the functioning of the series as a whole. In this paper we numerically implement the formula for simultaneous imaging and inversion.

We begin by using the numerical implementation to discuss two basic strategies. (1) We stabilize the addition of high-order terms with a weighted cutoff of high-frequency/high-wavenumber portions of the series, a strategy that bears a strong resemblance to the so-called truncated singular value decomposition inverse methods. This is found to be necesssary because of the reliance of the $n$ 'th term in the subseries' on the $n$ 'th derivative of the input. (2) We demonstrate a candidate approach to the compensation for bandlimited input (i.e., missing low and high frequencies), using the "gap-filling", or spectral extrapolation techniques that exist in the literature for bandlimited impedance inversion. Finally, we use the analogy of the Taylor's series expansion of an exponential function to predict and explain the close relationship between (a) the order of the imaged/inverted model reconstruction (i.e. the number of terms used in its computation), and (b) the frequency content of the reconstruction. This relationship, simply put, says that a low-order truncation of the series corresponds to a low-frequency (or smooth) reconstruction. Adding more terms adds higher frequencies; this suggests the existence of a trade-off between resolution and truncation order that may be critical in practical computation.

Simple acoustic 1D normal incidence configurations for the numeric examples are used in order to make simple comments on the broad nature of various subseries, and the computation of truncated versions thereof. We expect these comments, although based on simplistic examples, to apply in their essence (although no doubt with some variation in detail) to more complicated instances of inverse scattering series algorithms (i.e., multidimensional, elastic generalizations), because of their reliance on the same basic math-physics formalism.


## 1 Introduction: Models and Numerical Issues

In Innanen et al. (2004), we investigated the derivation and properties of the quantity

$$
\begin{equation*}
\alpha_{S I I}^{(n)}(z)=K_{n} \frac{d^{n}}{d z^{n}}\left(\int_{0}^{z} \alpha_{1}\left(z^{\prime}\right) d z^{\prime}\right)^{n}, \tag{1}
\end{equation*}
$$

where $\alpha_{1}(z)$ is the linear component of $\alpha(z)=1-c_{0}^{2} / c^{2}(z)$ for an acoustic, 1D normal incidence seismic problem, and

$$
\begin{equation*}
K_{n+1}=\frac{(-1)^{n}}{n+1}\left(\frac{1}{2^{n}}\right)^{2}\left[\sum_{k=0}^{n} \frac{1}{k!(n-k)!}\right] . \tag{2}
\end{equation*}
$$

In this section, we begin the investigation into the numerical computation of

$$
\begin{equation*}
\alpha_{S I I}(z)=\sum_{n=1}^{N} \alpha_{S I I}^{(n)}(z) . \tag{3}
\end{equation*}
$$

### 1.1 Synthetic 1D Models, Data and Born Approximations

In this section we use some simple Earth models to exercise the numerical computation of equation (3). We begin by generating 1D models, pathologically designed so that, at the last value of the discrete $z$ vector, the integral of $\alpha_{1}(z)$ is approximately zero, which avoids having the end of the signal behave like a strong reflector. This is done with no loss of generality, since any data set can have such an addendum included beyond the deepest point of interest. The first such model has 4 wavespeeds, the reference $c_{0}=1500 \mathrm{~m} / \mathrm{s}$, and $c_{1}=1600 \mathrm{~m} / \mathrm{s}, c_{2}=1650 \mathrm{~m} / \mathrm{s}, c_{3}=1467 \mathrm{~m} / \mathrm{s}$. (This last wavespeed value ensures that the above constraint holds.) These latter 3 wavespeeds correspond to layers which begin at depths 300 m , 500 m , and 700 m respectively.

This model is used to generate full-bandwidth data ( $D(z)$ at pseudo-depth $z$ ), then the Born approximation $\alpha_{1}(z)$, and its integral, which we refer to as $\mathcal{H}\left\{\alpha_{1}\right\}$. These are plotted respectively in Figure 1. This same process is carried out on a suite of Earth models, each having been chosen for (i) simplicity, and/or (ii) high-contrast, and/or (iii) somewhat complex structure. Table 1 details the models used.

### 1.2 Brute Implementation

We begin by "naive" application of the formula. The first term returns $\alpha_{1}$. The result of computing and adding-in the second and third terms is seen in Figure 2. The figure is organized as follows: the top panel (a) is the synthetic data; below this (b) consists of two functions, the Born approximation $\alpha_{1}(z)$ (dashed), and the true perturbation $\alpha(z)$ (dotted). The required tasks of the inverse series are clear: the inversion must correct the amplitudes, and the imaging must correct the locations. In other words, the inversion must make the dashed be the same as the dotted in the up-down direction of the plots, and the imaging must


Figure 1: Synthetic data (a) corresponding to Model 1 in Table 1 and its integrals; the Born approximation $\alpha_{1}(z)$ is in (b), and its integral $H\left\{\alpha_{1}\right\}$ is in (c). This last is the main ingredient in computing the coupled imaging and inversion. All three plots are against pseudo-depth z (m).

| Depth (m) | Model 1 ( $\mathrm{m} / \mathrm{s}$ ) | Model 2 $(\mathrm{m} / \mathrm{s})$ | Model 3 $(\mathrm{m} / \mathrm{s})$ | Model 4 $(\mathrm{m} / \mathrm{s})$ |
| :--- | :--- | :--- | :--- | :--- |
| $300-500$ | 1600 | 1600 | 2000 | 2000 |
| $500-700$ | 1650 | 1650 | 1700 | 2200 |
| $700-750$ | 1467 | 1600 | 1422 | 1423 |
| $750-800$ | - | 1570 | - | - |
| $800-870$ | - | 1530 | - | - |
| $870-910$ | - | 1500 | - | - |
| $910-\infty$ | - | 1454 | - | - |

Table 1: All Earth models used in the following imaging/inversion examples. All have reference media $(z<300 \mathrm{~m})$ characterized by wavespeed $c_{0}=1500 \mathrm{~m} / \mathrm{s}$. Model 2 has structure deeper than the others: the dash - signifies that the model remains constant at the last given wavespeed value. For instance, Model 1 is constant at $1467 \mathrm{~m} / \mathrm{s}$ below 700 m .
make the dashed be the same as the dotted in the left-right direction. The next panel (c) again illustrates $\alpha_{1}(z)$ (dashed), and includes the cumulative result of the terms in equation (3) beyond the first (solid). The lower panel (d) superimposes the full inversion results, $\alpha_{1}$ + cumulative result (solid), against the true perturbation $\alpha(z)$. In all figures of this kind that follow, "added value" associated with higher order terms in the series is demonstrated by having the plots in (d) become close to one another.

One can see the disturbances created by the series at the discontinuities - clearly more terms are needed to correct the location of the discontinuities. It is interesting to note that, by the third term, the amplitude correction (inversion) has come, visually, close to accomplishing its task; away from the discontinuities, and following the Born structure of the model, the layers have found their desired amplitude.


Figure 2: The third-order correction from equation (3), using Model 1 from Table 1. (a) Data input; (b) Born approximation (dashed) vs. true perturbation (dotted); (c) Born approximation (dashed) vs. secondorder correction (solid); (d) sum of Born approximation and correction (solid) vs. true perturbation. The inversion task is close to being done.

Continuing with the naive application of equation (3), compute the fourth term and add.

This is plotted in Figure 3. Clearly the discontinuities on the higher derivative operators quickly create large oscillations over the whole signal. Shortly hereafter the sum "blows up". Something more sophisticated is required.


Figure 3: The cumulative sum up to fourth order from equation (3), using Model 1 from Table 1. (a) Data input; (b) Born approximation (dashed) vs. true perturbation (dotted); (c) Born approximation (dashed) vs. second-order correction (solid); (d) sum of Born approximation and correction (solid) vs. true perturbation. Divergence occurs shortly hereafter.

### 1.3 Stabilizing the $n$ 'th Derivative

The uncontrolled oscillation of the unstable cumulative sum suggests that it is the highfrequency portions of the derivative operators which cause the problem. A simplistic regularization, amounting to a smoothed cutting-off of the highest frequencies, allows resolution to be traded-off for stability. The regularization works as follows. Windows in the $k$ domain are constructed by convolving a gate function with a Gaussian; such a window is therefore defined with two parameters, the variance of the Gaussian, and the width of the gate.

The latter controls the frequency cut-off, and the former controls the smoothness of the "shoulders" of the cutoff. Figure 4 illustrates these windows.


Figure 4: Four plots of regularization windows in the $k$ domain for various parameters. A low variance Gaussian convolved with narrow gate is illustrated (a) vs. wide gate (b) as is a high variance Gaussian convolved with narrow gate (c) vs. wide gate (d). Character of the ensuing regularization may be altered by changing these parameters to suit specific situations.


Figure 5: Regularized derivative operators in the $k$ domain (solid) vs. un-regularized operators (dotted). (a) $d / d z$, (b) $d^{2} / d z^{2}$, (c) $d^{3} / d z^{3}$, and (d) $d^{4} / d z^{4}$. "Levels" of regularization are dictated by the window parameters (see Figure 4).

In these numerical examples the derivatives are computed in the wavenumber domain. Figure 5 illustrates the un-regularized (dotted) and regularized derivative operators (solid) for orders 1 through 4.

Since the $n$ 'th derivative is a linear, space invariant operation, its Fourier domain representation is equivalent to a singular value expansion. A weighted suppression of the Fourier coefficients of the operator at high frequency is therefore equivalent to stabilization via truncated singular value decomposition, in the parlance of linear inverse theory.

### 1.4 Numerical Examples of Simultaneous Imaging and Inversion

Using a sharp truncation (i.e. low variance) and a window that cuts off the high-frequency portion of the derivative operators on either end of the spectrum, we proceed to compute $\approx 100$ terms in the series of equation (3) for the same input. The results are in Figure 6. The regularization parameters must be chosen for each example; we did this by trial and error for these examples.

Clearly much of the character of the true model is captured here - see the additional examples in Figures 7 -9. The main deviation is in the large contrast examples, in which inaccuracy


Figure 6: The cumulative sum of $\approx 100$ terms from equation (3), using Model 1 from Table 1. (a) Data input; (b) Born approximation (dashed) vs. true perturbation (dotted); (c) Born approximation (dashed) vs. (solid); (d) sum of Born approximation and correction (solid) vs. true perturbation. Using a low-variance Gaussian and a gate that cuts the derivative operator off on each end of the spectrum, 100 terms from equation (3) are computed and summed. Results capture closely the desired result.
in the imaging (reflector location) appears. In these examples, as evidenced by the poorer resolution, the derivative operators must also be more aggressively regularized (truncated); however, the missing bandwidth does not explain the inaccuracy. It is reasonable to postulate that it is due to the missing higher order imaging terms in the approximation; indeed there was earlier evidence that the partial nature of this scheme meant inaccuracy at higher reflection coefficients.


Figure 7: The cumulative effects of $\approx 100$ terms from equation (3), using Model 2 from Table 1. (a) Data input; (b) Born approximation (dashed) vs. true perturbation (dotted); (c) Born approximation (dashed) vs. correction (solid); (d) sum of Born approximation and correction (solid) vs. true perturbation. Using a low-variance Gaussian and a gate that cuts the derivative operator off on each end of the spectrum, 100 terms from equation (3) are computed and summed. Results capture closely the desired result.


Figure 8: The cumulative effects of $\approx 100$ terms from equation (3), using Model 3 from Table 1. (a) Data input; (b) Born approximation (dashed) vs. true perturbation (dotted); (c) Born approximation (dashed) vs. correction (solid); (d) sum of Born approximation and correction (solid) vs. true perturbation. Using a low-variance Gaussian and a gate that cuts the derivative operator off on each end of the spectrum, 100 terms from equation (3) are computed and summed. Results capture closely the desired result, but some inaccuracy in the high-contrast correction is noticed.


Figure 9: The cumulative effects of $\approx 100$ terms from equation (3), using Model 4 from Table 1. (a) Data input; (b) Born approximation (dashed) vs. true perturbation (dotted); (c) Born approximation (dashed) vs. correction (solid); (d) sum of Born approximation and correction (solid) vs. true perturbation. Using a low-variance Gaussian and a gate that cuts the derivative operator off on each end of the spectrum, 100 terms from equation (3) are computed and summed. Results capture closely the desired result, but greater inaccuracy in the high-contrast correction is noticed.

## 2 Noise and Bandlimitation

In this section we consider two key departures of real-world problems of seismic processing from theory: noise and bandlimitation.

### 2.1 Robustness to Incoherent Noise

It has been mentioned that, in principle, equation (3) should be unstable, and, indeed, it has been found that the derivative operators require high-frequency truncation, more-so with greater model contrast. Once this has been accomplished the recovered models capture the sharpness of the contrasts admirably, through the computation of $\approx 100$ terms. These results apply for full bandwidth data with no noise. In the next section we will address the problem of bandlimitation. Here we consider the results in the presence of varying levels of additive incoherent (Gaussian) noise.

Consider the example of Figure 7. Three realizations of Gaussian noise (with variance of approximately $\% 1$ of the data amplitudes) are added to the data, and the imaging/inversion terms are recomputed using the $\int_{0}^{z} \alpha_{1}\left(z^{\prime}\right) d z^{\prime}$ that comes from integrating the noisy data. Figures $10-12$ contain the results.

The results are clearly deteriorated by the presence of noise - in fact, realization to realization the same level of noise can produce quite different results. Both the location and the amplitude of the correction are corrupted, but a qualitative conclusion is that the inversion, or amplitude results are the most sensitive. In all cases the imaging component, i.e. the movement of the reflectors, still marks a great improvement over the Born approximation.

The quality of these Earth models, recovered in the presence of noise, is heartening in the sense that one might expect the smallest amount of noise to render computation of equation (3) completely unstable. Nevertheless, it is clear that a very high-fidelity estimate of $\alpha_{1}$ will be of tremendous value. The estimation of such an input might include edge-preserving noise reduction strategies, as well as (presumably) low frequency/wavenumber filling strategies.


Figure 10: The data used to generate Figure 7 is corrupted with $\% 1$ noise, and the imaging/inversion results are recomputed. Organization of results is as in Figure 7. This is the first of three realizations.


Figure 11: The data used to generate Figure 7 is corrupted with $\% 1$ noise, and the imaging/inversion results are recomputed. Organization of results is as in Figure 7. This is the second of three realizations.


Figure 12: The data used to generate Figure 7 is corrupted with $\% 1$ noise, and the imaging/inversion results are recomputed. Organization of results is as in Figure 7. This is the third of three realizations.

### 2.2 Spectral Extrapolation: Computing $\alpha_{1}$ from Bandlimited Data

There are some striking computational similarities between these inverse scattering subseries methods and other well-studied problems of inversion of seismic wave field measurements. In particular, at its core are integral operations on a quantity which ideally should be realized over a full band of frequencies, including a DC component. Since this quantity is based on wave field measurements which are unavoidably bandlimited, however, it too is bandlimited in practice. The consequences to the output of the subseries requires study.

As with seismic inverse methods such as impedance inversion, methods for extension of the spectrum to zero frequency do exist. In this section an investigation of the applicability of such methods to our various subseries, in their current 1D incarnations, is presented. We show some specific results and their application to examples akin to those previously presented; but the idea here is to broadly propose the use of a framework, long in the literature (see below for references), for tackling the important problem of bandlimited inversion.

A measured signal, wavelet deconvolved, may be assumed to be the bandlimited expression of some reasonably simple full-bandwidth signal type. An example might be a series of lagged and scaled delta functions, i.e. a reflectivity series. The spectrum of the measured signal which is the full bandwidth spectrum multiplied by a gate function - can be extended, in principle uniquely, guided by this assumption.

The calculation of $\alpha_{1}$ is linear inversion, and with the right assumptions it is equivalent to 1D inversion for acoustic impedance. So the root of the low-frequency trouble is in essence the same for the inverse series as it is for impedance inversion; strategies for coping with this lack in impedance recovery may therefore by readily applicable in the 1D normal incidence inverse series examples presented here.

## Basic Concepts of Bandlimited Impedance Inversion

We begin by discussing spectral extrapolation proper, as part of a brief review of methods for bandlimited inversion. A delta function in the time domain has a complex sinusoid as a spectrum. The frequency of this sinusoid is uniquely determined by the lag of the delta function. Since a sinusoid with additive noise is adequately described with autoregressive (and correctly modelled with ARMA) models, a spectrum of this kind that has been bandlimited can be effectively predicted and therefore extended beyond the bandwidth. Larger order ARMA and AR models may, further, predict the summation of many sinusoids (of different frequencies), and therefore extend the spectra of bandlimited delta series with many varied lags. Consider the reflectivity series that is this sum of delta functions, following Walker and Ulrych (1983):

$$
\begin{equation*}
r(t)=\sum_{k} b_{k} \delta(t-k \Delta), \tag{4}
\end{equation*}
$$

where $b_{k}$ are the reflectivity coefficients and $\Delta$ is the time interval of the experiment. The Fourier transform, or the spectrum of the reflectivity, $R(f)$, is:

$$
\begin{align*}
R(f) & =\int_{-\infty}^{\infty} e^{-i 2 \pi f t}\left[\sum_{k} b_{k} \delta(t-k \Delta)\right] d t  \tag{5}\\
& =\sum_{k} b_{k} e^{-i 2 \pi f k}
\end{align*}
$$

if $\Delta$ is set to unity. So the $R(f)$ is this sum of weighted complex sinusoids. With a finite bandwidth, the data $R(f)$ may be extended to the lower (and, to some extent, the higher) frequencies via a prediction scheme. Although the noisy sum of sinusoids is correctly modelled via an ARMA model, Walker and Ulrych (1983) recommend, rather, a truncated AR prediction approach as being more stable.

An autoregressive process has a predictable part and an unpredictable, or innovational, part. That is, the $j^{\prime}$ th element of a series $y_{j}$ can be written as a linear combination of previous elements, plus an error (innovation) term:

$$
\begin{equation*}
y_{j}=\sum_{k=1}^{p} a_{k} y_{j-k}+e_{j}, \tag{6}
\end{equation*}
$$

or

$$
\begin{equation*}
y_{j}=-\sum_{k=1}^{p} g_{k} y_{j-k}+e_{j}, \tag{7}
\end{equation*}
$$

where $g_{k}=-a_{k}$ and $g_{0}=1$ are the coefficients of the prediction error filter. With knowledge of the variance of the error $e_{j}\left(\sigma_{e}\right)$, and having in some appropriate way computed the autocorrelation matrix of $y_{j}\left(\mathbf{R}_{\mathbf{y y}}\right)$, the prediction error filter $\mathbf{g}$ is found by solving the system

$$
\begin{equation*}
\mathbf{R}_{\mathrm{yy}} \mathbf{g}=\sigma_{e}^{2} \mathbf{i} \tag{8}
\end{equation*}
$$

where $\mathbf{i}=[1,0,0, \ldots, 0]^{T}$ and $\mathbf{g}$ and $\mathbf{R}_{\mathbf{y y}}$ etc. all have complex elements.
Associating bandlimited $R(f)$ with $y_{j}$, the filter $\mathbf{g}$ may be applied to existing values of $R(f)$ to produce an extended spectrum data point, one discrete frequency step closer to zero. This point is thereafter treated as an existing value of the reflectivity spectrum, and the process is repeated.

Spectral extension, which is essentially extrapolation via a difference equation, has a growing error in practical application as it predicts further from the known data values. There is in fact an overall tendency of extended spectra to decay. This can be avoided by an efficient
alternate form of prediction based on the gap-filling algorithms of Wiggins (1972), in which the existence of known data values on both sides of the unknown region is utilized to avoid such accumulating error. Since, by conjugate symmetry, the negative frequencies are also known, the missing lower frequencies may be thought of as such a gap, on either side of which are known data. An unknown point in the spectrum is considered to be the weighted sum of predictions from the "left" and from the "right" data sets (and in which the weights are determined by minimizing the overall prediction error). Walker and Ulrych (1983) and Ulrych and Walker (1984) show that a scheme of this kind finds all the missing data points (of the discrete spectrum) at once. One may extend to high frequencies in the same way, although, since the high frequency gap is considerably larger than the low frequency one, the predictions will be more highly attenuated.

The problem may, secondly, be addressed by utilizing the formalism of inverse theory (Oldenburg, 1984; Oldenburg et al., 1983); the approach adopted here is the work of Ulrych (1989), which uses an entropic norm to solve for the whole spectrum with the existing spectrum as the input. The bandlimited signal is viewed as being non-unique, in the sense that an infinite number of models (i.e. time-domain signals) may be Fourier transformed and bandlimited to produce the data (i.e. the bandlimited Fourier transform). The method incorporates prior information garnered from the data (usually a threshold) as a constraint, which directs the model "choice", from the set of allowable models, to be that which is sparse, or spike-like. In such models the low and high frequencies tend to be present.
Either of the two approaches discussed above is well-suited to the task of estimating the expected form of the full-bandwidth Born approximation. In the examples to follow, the latter is used, simply because it tends to estimate the high frequencies of the spectrum with greater accuracy.

## Numerical Examples Using Bandlimited Data

In Figure 13, detail of the filled data is given close to a single reflection. The top panel (a) is the full bandwidth data, the second panel (b) is the bandlimited equivalent (assuming $\Delta t=0.004 \mathrm{~s}$, these examples correspond to a band of $10-50 \mathrm{~Hz}$ ), the third is the spectrallyextrapolated output, and the fourth is a detail (zoom) of the extrapolated (solid) pulse vs. the full bandwidth pulse (dashed).
In Figure 14, the results of Figure 7 are re-computed using spectrally-extrapolated estimates of the data, with only bandlimited traces as input. Further panels are added to the top of this figure; first is the bandlimited data, below that is the filled data, and below that are the panels illustrating the inversion/imaging as previously used.
In Figures 14 and 15, the relatively spatially-complex data example and the high-contrast data example, both bandlimited are shown to respond very well to the spectral-extrapolation pre-processing.
Of course, these examples have not specifically pushed the limits of spectral extrapolation technology. The key is to, in some quantitative way, address the issue of bandlimitation
in these non-linear inversion schemes. The value of these examples is in the demonstration of working methods for compensating, often via some reasonable prior knowledge, for imperfections of the seismic experiment.


Figure 13: Detail of the filled data is given close to a single reflection. (a) full bandwidth data, (b) the bandlimited equivalent, (c) spectrally-extrapolated output, and (d) detail of the extrapolated (dashed) pulse vs. the full bandwidth pulse (solid).


Figure 14: Simultaneous imaging and inversion as applied to Model 2 from Table 1 is repeated on spectrallyextrapolated estimates of the data, using only bandlimited traces as input. Further panels are added to the top of this figure (a) the input bandlimited data, (b) the spectrally-extrapolated (recovered) full-bandwidth data, (c) Born approximation (dashed) vs. true perturbation (dotted); (d) Born approximation (dashed) vs. second-order correction (solid); (e) sum of Born approximation and correction (solid) vs. true perturbation.


Figure 15: Simultaneous imaging and inversion as applied to Model 4 (high contrast) from Table 1 is repeated on spectrally-extrapolated estimates of the data, using only bandlimited traces as input. Further panels are added to the top of this figure (a) the input bandlimited data, (b) the spectrally-extrapolated (recovered) fullbandwidth data, (c) Born approximation (dashed) vs. true perturbation (dotted); (d) Born approximation (dashed) vs. second-order correction (solid); (e) sum of Born approximation and correction (solid) vs. true perturbation.

## 3 The Relationship Between Order and Frequency Content

In low-order truncations of the simultaneous imaging and inversion subseries there occurs an "explosion" in the depth domain - uncontrolled high-amplitude oscillation of the reconstruction (see for example Figure 3). After the addition of many terms, this oscillation can be seen to "settle down" upon the desired reconstruction. (This characteristic evolution-with-order is also true of the leading order imaging subseries.) It is tempting to ignore this behaviour prior to convergence, since it appears to be uninterpretable. Here we investigate the numerical convergence issues of the simultaneous imaging and inversion subseries using some simple Taylor's series examples as a guide. The result is an analytical/numerical framework within which to better understand the seemingly unstable order-by-order behaviour of subseries that locate reflectors.

We know that the $n$ 'th term in the simultaneous imaging and inversion subseries (and the leading order imaging subseries) involves the $n$ 'th spatial derivative of an input. We may therefore make the very loose mathematical comment that, in the wavenumber domain, terms in the series will involve increasing powers of $(i k)$ :

$$
\begin{equation*}
\alpha_{S I I}(k) \approx \ldots+(i k C)^{n} G(k)+\ldots \tag{9}
\end{equation*}
$$

and also some function $G(k)$, and some constant $C$. Since we have chosen the true model in these tests, we also know something about what we are constructing; namely, discontinuous functions that correct spatial locations of reflectors and their amplitudes (in this case the discontinuous functions are Heaviside functions). In the wavenumber domain, then, the reconstruction will be an exponential function with a wavenumber-dependent weight, so again loosely:

$$
\begin{equation*}
\alpha_{S I I}(k) \approx F(k) e^{i k C} \tag{10}
\end{equation*}
$$

where $F(k)$ is the weighting function.
In other words, the simultaneous imaging and inversion process is very similar to the Taylor's series expansion of an exponential function about zero, i.e. involving an infinite series of polynomials:

$$
\begin{equation*}
e^{x}=1-x+\frac{1}{2} x^{2}-\frac{1}{3!} x^{3}+\ldots, \tag{11}
\end{equation*}
$$

where we interpret the argument $x$ in our case as being $i k C$.
Since our imaging/inversion goal is to construct these corrective, discontinuous signals numerically over a finite wavenumber interval, the correct analogy is that of constructing $e^{x}$ over some fixed interval $x=\left(0, x_{\max }\right)$. In Figure 16, we estimate the (better behaved) function $e^{-x}$ over such a fixed interval for a number of orders.

The evolution of the approximation with truncation order is straightforward: for each successive term added, the approximation is made accurate (i.e. it converges) over a larger interval of $x$. Beyond this region of convergence the approximation diverges at a rate of $x^{n}$ for a truncation at the $n$ 'th term.


Figure 16: Taylor's series approximation of $e^{-x}$ over a fixed interval; six approximations, 0 'th-3'rd, followed by 9 'th and 17 'th. As more terms are added, the approximations converge over a lengthening interval of $x$.

Applying this thinking to the simultaneous imaging and inversion subseries (and by association the general nature of the inverse scattering series terms for processing and inversion of primaries), we recognize that there is an implied relationship between the convergent bandwidth of the reconstruction and the order at which the series construction is truncated. To wit: since the interval of construction (of exponential-like models) is in the $k$ domain, we can expect the series to converge, with added terms, over a growing wavenumber interval. To demonstrate, we include in Figures $17-21$ numeric examples of the reconstruction of $\alpha_{\text {SII }}$ in the $k$ domain (Figure 17a etc.). Although the signal in the $k$ domain is somewhat more complicated than $e^{-x}$ in the $x$ domain, the evolution across increasing truncation orders is seen to behave very similarly, with an increasing interval of convergence in $k$. As a final
step for each example (i.e. after each summation is complete), we manually suppress the divergent frequencies from each example, inverse Fourier transform, and display the result (Figures 17b, c - 21b, c). The reconstructions go from smooth to sharp as we move from low-order to high-order truncation.

The direct relationship between the frequency content of the reconstruction is in this way made more apparent to the eye (more apparent, that is, than if we had left in the dominant but uninformative divergent portions of the spectra) and the approximation order of the model. This insight may be of use as it implies a possible tradeoff that can be made between the resolution of the reconstruction and the number of terms used (which in more realistic instances could have very important computational consequences). It has not escaped the authors' attention that the very bandwidth extrapolation methods discussed in this paper (to deal with missing low frequency data) may be applicable in such series computations to reduce the number of terms needed: i.e., the series terms might be used up to a certain order, beyond which the remaining signal on the $k$ interval may be "predictable" in the sense of bandwidth extrapolation. This remains speculative.


Figure 17: Truncated computation of $\alpha_{S I I}(z)$, order 5: (a) in the wavenumber domain, the construction appears to diverge rapidly with larger $k$; (b) and (c) manually suppressing the divergent wavenumbers and inverse Fourier transforming, we see that the correction is present but highly smoothed when compared to the true perturbation (dotted) and the input Born approximation (dashed).


Figure 18: Truncated computation of $\alpha_{S I I}(z)$, order 10: (a) in the wavenumber domain, the construction appears to diverge rapidly with larger $k$, although less so at lower $k$ than for order 5; (b) and (c) manually suppressing the divergent wavenumbers and inverse Fourier transforming, we see that the correction is somewhat less smoothed when compared to the true perturbation (dotted) and the input Born approximation (dashed).


Figure 19: Truncated computation of $\alpha_{S I I}(z)$, order 20: (a) in the wavenumber domain, the construction can now be seen to be non-increasing over a wider interval of $k$ than in the 5 'th and 10 'th order truncations; (b) and (c) manually suppressing the divergent wavenumbers and inverse Fourier transforming, we see that the correction is concurrently becoming "sharper" when compared to the true perturbation (dotted) and the input Born approximation (dashed).


Figure 20: Truncated computation of $\alpha_{S I I}(z)$, order 40: (a) in the wavenumber domain, the construction can now again be seen to be non-increasing over a wider interval; (b) and (c) manually suppressing the divergent wavenumbers and inverse Fourier transforming, we see again that the correction is concurrently becoming "sharper" when compared to the true perturbation (dotted) and the input Born approximation (dashed).


Figure 21: Truncated computation of $\alpha_{S I I}(z)$, order 80: (a) in the wavenumber domain, the construction can now again be seen to be non-increasing over a wider interval; (b) and (c) manually suppressing the divergent wavenumbers and inverse Fourier transforming, we see again that the correction is concurrently becoming "sharper" when compared to the true perturbation (dotted) and the input Born approximation (dashed).

## 4 Conclusions

We consider some basic numeric issues associated with the computation and interpretation of a subseries that is involved with both imaging and inversion of seismic primaries. The subseries and the computations are concerned with the simple 1D normal incidence acoustic problem.

The raw computation of truncated versions of the subseries are shown to produce unstable results, a problem that may be overcome through the use of approximate derivative operators whose highest frequency components are suppressed. The operator approximations are designed to accommodate differing contrasts in the data and - we show - incoherent noise.

We also demonstrate the use of a form of spectral extrapolation that exists in the literature ostensibly for bandlimited impedance inversion. This methodology, on the assumption of delta-like data events, extrapolates from existing bandwidth intervals to missing intervals. We show with some simple examples the application of this approach to the computation of the Born approximation - filling in the low frequencies of the linear result provides a full-bandwidth input to the higher order terms in the series.

Finally, we consider the numerics of truncated model estimates from the imaging-inversion subseries; we demonstrate the relationship between truncation order and frequency/wavenumber content of the constructed model using a simple Taylor's series expansion of an exponential function as a guide.

The purpose of this work is to sketch out some of the basic numerical issues at the core of the inverse scattering subseries' for the imaging and inversion of primaries. The expectation is that some or all of the issues encountered here will be visited on inverse scattering seriesbased computations in higher dimensions and for more complicated models; further, that some of the strategies developed for these simple instances might be extended (also to higher dimensions and greater complexity).

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[^0]:    ${ }^{1}$ Operators without hats in this paper are in the displacement space, those with hats are in PS space.

[^1]:    ${ }^{1}$ We sometimes refer to wavenumbers $k_{1}$ and $k_{2}$ as "frequencies", referring to the simple relationship with $f$ as in $k_{1}=2 \pi f_{1} / c_{0}$. Also, usefully, ratios of frequency-related quantities are equivalent: $f / f_{r}=\omega / \omega_{r}=k / k_{r}$.

[^2]:    ${ }^{1}$ It is reminiscent of the convention of discussing convolutions as $h\left(z^{\prime}\right)=f\left(z^{\prime}\right) * g\left(z^{\prime}\right)$, i.e. with seemingly careless use of the output variable $z^{\prime}$ in both input functions.

[^3]:    ${ }^{2}$ We will use the terms primary and secondary a lot in this section. The former shouldn't be confused with primaries as in "the primaries and multiples of seismic data". In this paper, we use the word to mean those specific portions of the inverse scattering series defined here.

